MAT2400: Mandatory assignment #1, Spring 2020

Suggested solution

Problem 1.

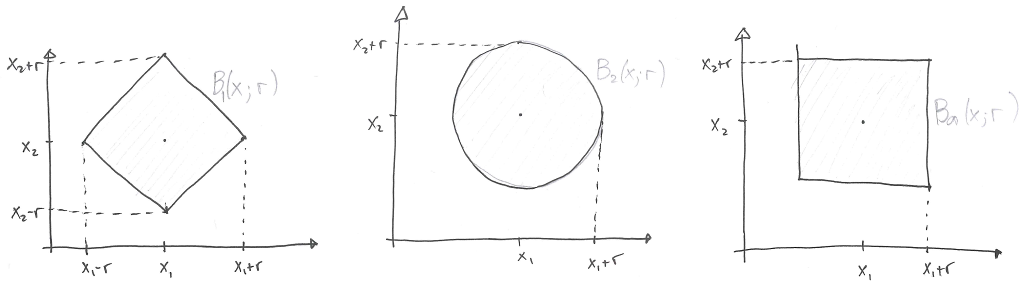


FIGURE 1. The balls $B_i(x;r)$ for $i = 1, 2, \infty$. The ball contains everything within the indicated area, but not the boundary.

Problem 2.

(a) Let $x \in \mathbb{R}^2$ and let $r > 0$. From the hint we know that

$$d_1(x,y) \leqslant \sqrt{2} d_2(x,y) \qquad \text{and} \qquad d_2(x,y) \leqslant d_1(x,y) \qquad \forall\, x,y \in \mathbb{R}^2.$$

Hence, if $s = \frac{1}{\sqrt{2}} r$ and $t = r$ then for any $y \in \mathbb{R}^2$,

$$d_2(x,y) < s \quad \Rightarrow \quad d_1(x,y) \leqslant \sqrt{2} d_2(x,y) < \sqrt{2} s = r$$

and

$$d_1(x,y) < t \quad \Rightarrow \quad d_2(x,y) \leqslant d_1(x,y) < t = r$$

which means the same as

$$B_2(x;s) \subset B_1(x;r) \qquad \text{and} \qquad B_1(x;t) \subset B_2(x;r).$$

(b) It is enough to find an open ball in one of the metric space into which we cannot fit an open ball from the other metric space. Let $r = 1/2$, let $x \in \mathbb{R}^2$ be arbitrary and consider

$$B_d(x;r) = \{y \in \mathbb{R}^2 \ : \ d(x,y) < 1/2\} = \{x\}$$

(since $d(x,y)$ is less than $1/2$ only when $x = y$). No open ball in $(\mathbb{R}^2, d_2)$ can fit inside $\{x\}$, since if $t > 0$ is any number then $B_{d_2}(x;t)$ contains infinitely many points, but $B_d(x;r)$ doesn't.

(c) Let $V \subset X$ be open in (X, ρ). Then for every $x \in V$ we can find $r > 0$ such that $B_\rho(x;r) \subset V$. Let $t > 0$ be such that $B_\gamma(x;t) \subset B_\rho(x;r)$; then in particular, $B_\gamma(x;t) \subset V$. Hence, V is also open in (X, γ). The opposite follows by symmetry.

(d) We only prove "$\Rightarrow$", since "$\Leftarrow$" follows by symmetry. Assume that $\{x_n\}_n$ converges to x in (X, ρ). Let $\varepsilon > 0$. Since ρ and γ are equivalent, there is some $\tilde{\varepsilon} > 0$ such that $B_\rho(x;\tilde{\varepsilon}) \subset B_\gamma(x;\varepsilon)$. Let $N \in \mathbb{N}$ be such that $\rho(x, x_n) < \tilde{\varepsilon}$ for all $n \geqslant N$ – that is, $x_n \in B_\rho(x;\tilde{\varepsilon})$. But then also $x_n \in B_\gamma(x;\varepsilon)$ – that is, $d(x, x_n) < \varepsilon$ for all $n \geqslant N$. Hence, $\{x_n\}_n$ converges to x in (X, γ).

(e) Let $K \subset X$ be a compact subset for (X, ρ). For any sequence $\{x_n\}_{n \in \mathbb{N}}$ in K, let $\{x_{n_k}\}_{k \in \mathbb{N}}$ be a convergent subsequence in (X, ρ), converging to a point $a \in K$. By (d), the subsequence also converges to a in (X, γ). Hence, K is also compact in (X, γ). The converse follows by symmetry.

(f) Let us write $\gamma(x,y) = \phi(\rho(x,y))$, as in the hint.

(i) Since $\gamma(x,y) < 1$ for all $x, y \in X$, we have $X = B_\gamma(x;1)$ for any $x \in X$, so X is bounded.

(ii) γ is clearly nonnegative and symmetric, and vanishes iff $x = y$ (since $\phi(s) = 0$ only when $s = 0$). For the triangle inequality, we prove first that ϕ is a *subadditive* function, that is, $\phi(s+t) \leqslant \phi(s) + \phi(t)$. Indeed,

$$\phi(s+t) = \frac{s+t}{1+s+t} = \frac{s}{1+s+t} + \frac{t}{1+s+t} \leqslant \frac{s}{1+s} + \frac{t}{1+t} = \phi(s) + \phi(t)$$

where the inequality follows from the fact that the denominators are smaller. Moreover, ϕ is an increasing function, as $\phi'(s) = \frac{1}{(1+s)^2} \geqslant 0$. It now follows that

$$\begin{aligned}
\gamma(x,y) &= \phi(\rho(x,y)) \\
&\leqslant \phi\big(\rho(x,z) + \rho(z,y)\big) && \text{(since } \phi \text{ is increasing)} \\
&\leqslant \phi(\rho(x,z)) + \phi(\rho(z,y)) && \text{(since } \phi \text{ is subadditive)} \\
&= \gamma(x,z) + \gamma(z,y).
\end{aligned}$$

(iii) Since $\gamma(x,y) \leqslant \rho(x,y)$ for all x, y, we have $B_\rho(x;r) \subset B_\gamma(x;r)$ for all $x \in X$, $r > 0$. Conversely, let $r > 0$ and $x \in X$. Set $t = \phi(r)$. If $y \in X$ is such that $\gamma(x,y) < t$ then, by definition, $\phi(\rho(x,y)) < t = \phi(r)$, so necessarily $\rho(x,y) < r$ (this follows from the fact that ϕ is strictly increasing: If $\phi(s) < \phi(r)$ then $s < r$). This proves that $B_\gamma(x;t) \subset B_\rho(x;r)$.

Problem 3.

(a) We can write

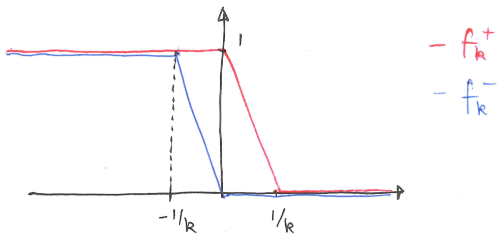
$$\mathbb{1}_{(-\infty,0]}(y) + kd(x,y) = \begin{cases} 1 + k|x-y| & \text{if } y \leqslant 0 \\ k|x-y| & \text{if } y > 0. \end{cases}$$

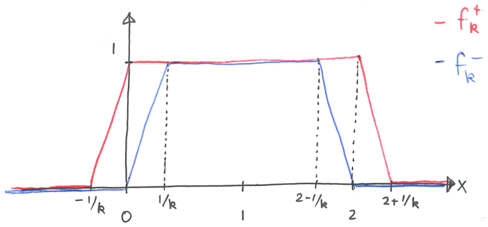
We split into three cases: If $x \geqslant 0$ then $1 + k|x-y| \geqslant 1$ and $k|x-y| \geqslant 0$, with equality attained in the limit $y \to x$. Hence, $f_k^-(x) = 0$. If $x \leqslant -1/k$ then $1 + k|x-y| \geqslant 1$, with equality when $y = x$, while $k|x-y| \geqslant k|x-0| = 1$. Hence, $f_k^-(x) = 1$. Last, if $-1/k < x < 0$ then $1 + k|x-y| \geqslant 1 + k|x-0| = 1$, while $k|x-y| \geqslant k|x-0| = k|x| = -kx < 1$. We conclude that

$$f_k^-(x) = \begin{cases} 1 & \text{if } x \leqslant -1/k \\ -kx & \text{if } -1/k < x < 0 \\ 0 & \text{if } x \geqslant 0. \end{cases}$$

A similar argument shows that

$$f_k^+(x) = \begin{cases} 1 & \text{if } x \leqslant 0 \\ 1 - kx & \text{if } 0 < x < 1/k \\ 0 & \text{if } x \geqslant 1/k. \end{cases}$$





(b)

$$f_k^-(x) = \inf_{y \in X} \big(f(y) + kd(x,y)\big) \leqslant \big(f(y) + kd(x,y)\big)\Big|_{y=x} = f(x),$$

$$f_k^+(x) = \sup_{y \in X} \big(f(y) - kd(x,y)\big) \geqslant \big(f(y) - kd(x,y)\big)\Big|_{y=x} = f(x).$$

(c)

$$f_k^-(x) = \inf_{y\in X}\big(f(y) + kd(x,y)\big) \geqslant \inf_{y\in X}\big(a + kd(x,y)\big) = a,$$

and from (a),

$$f_k^-(x) \leqslant f(x) \leqslant b.$$

The same argument applies to f_k^+.

(d) For a number $\varepsilon > 0$, let $z \in X$ be such that $f_k^-(x_2) \geqslant f(z) + kd(x_2, z) - \varepsilon$. We get

$$\begin{aligned}
f_k^-(x_1) - f_k^-(x_2) &= \inf_{y_1\in X}\big(f(y_1) + kd(x_1,y_1)\big) - \inf_{y_2\in X}\big(f(y_2) + kd(x_2,y_2)\big) \\
&\leqslant \inf_{y_1\in X}\big(f(y_1) + kd(x_1,y_1)\big) - \big(f(z) + kd(x_2,z) - \varepsilon\big) \\
&\leqslant \big(f(y_1) + kd(x_1,y_1)\big)\Big|_{y_1=z} - \big(f(z) + kd(x_2,z) - \varepsilon\big) \\
&= f(z) + kd(x_1,z) - f(z) - kd(x_2,z) + \varepsilon \\
&= k\big(d(x_1,z) - d(x_2,z)\big) + \varepsilon \\
&\leqslant kd(x_1,x_2) + \varepsilon
\end{aligned}$$

Switching the roles of x_1 and x_2 gives

$$\big|f_k^-(x_1) - f_k^-(x_2)\big| \leqslant kd(x_1,x_2) + \varepsilon.$$

Since $\varepsilon > 0$ was an arbitrary number, we get

$$\big|f_k^-(x_1) - f_k^-(x_2)\big| \leqslant kd(x_1,x_2),$$

and hence, f_k^- is Lipschitz continuous with Lipschitz constant no larger than k.

(e) Since f is bounded, there is some $M > 0$ such that $|f(x)| \leqslant M$ for all $x \in X$. Let $x \in X$, let $\varepsilon > 0$ and find $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ when $d(x,y) < \delta$. We now write

$$f_k^-(x) = \inf_{y\in X}\big(f(y) + kd(x,y)\big) = \min(T_1,\ T_2),$$

where

$$T_1 = \inf_{y\in B(x;\delta)}\big(f(y) + kd(x,y)\big), \qquad T_2 = \inf_{y\notin B(x;\delta)}\big(f(y) + kd(x,y)\big).$$

If we let $k \in \mathbb{N}$ be large enough that $k\delta \geqslant 2M$ then

$$T_2 = \inf_{y\notin B(x;\delta)}\big(f(y) + kd(x,y)\big) \geqslant \inf_{y\notin B(x;\delta)}\big(-M + kd(x,y)\big) \geqslant -M + k\delta \geqslant M.$$

But from (b) we know that $f_k^-(x) \leqslant f(x) \leqslant M$, so T_2 cannot possibly be smaller than T_1. Hence,

$$0 \leqslant f(x) - f_k^-(x) = f(x) - \inf_{y\in B(x;\delta)}\big(f(y) + kd(x,y)\big) = \sup_{y\in B(x;\delta)}\big(\underbrace{f(x) - f(y)}_{<\varepsilon}\underbrace{-kd(x,y)}_{\leqslant 0}\big) \leqslant \varepsilon,$$

so in particular,

$$|f(x) - f_k^-(x)| \leqslant \varepsilon.$$

(f) If f is uniformly continuous then the choice of δ in (e) is independent of the choice of $x \in X$. Hence, the argument in (e) works for any $x \in X$, and as a consequence, the conclusion $|f(x) - f_k^-(x)| < \varepsilon$ is true for any $x \in X$.