February 18, 2020

MAT2400 – Real Analysis

Mandatory assignment 1 of 2

Submission deadline

Thursday 27 February 2020, 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between writing in English or Norwegian. You will find a Norwegian–English mathematical dictionary at

uio.no/studier/emner/matnat/math/MAT2400/v20/norsk-engelsk-ordliste.html.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible; please use either the “Color” or “Grayscale” settings. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Note: You will find a list of hints on page 4.

Problem 1. Consider the set $X = \mathbb{R}^2$. For each of the following metrics on X, draw a sketch of the ball $B(x;r)$, where $x \in X$ and $r > 0$ are arbitrary. Indicate x and r clearly in your sketch.

(a) $d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|$

(b) $d_2(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$

(c) $d_\infty(x,y) = \max(|x_1 - y_1|, |x_2 - y_2|)$

Problem 2. Consider first the following definition:

Definition. Let X be a set and let ρ and γ be two metrics on X. We say that ρ and γ are *equivalent* if the open balls in (X,ρ) and (X,γ) are "nested". More precisely, ρ and γ are equivalent if for every $x \in X$ and $r > 0$ there exist $s, t > 0$ such that both

$$B_\rho(x;s) \subset B_\gamma(x;r) \qquad \text{and} \qquad B_\gamma(x;t) \subset B_\rho(x;r)$$

where $B_\rho(x;r) = \{y \in X \mid \rho(x,y) < r\}$ and $B_\gamma(x;r) = \{y \in X \mid \gamma(x,y) < r\}$.

(a) Prove that d_1 and d_2 are equivalent metrics on $X = \mathbb{R}^2$. (Draw a picture first!)

(b) Prove that d_2 (or d_1, for that matter) is *not* equivalent to the discrete metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \qquad \text{for } x, y \in \mathbb{R}^2.$$

For the rest of this problem we let X be some arbitrary set (not necessarily $\mathbb{R}^2$).

(c) Prove that if ρ and γ are equivalent metrics on X then (X,ρ) and (X,γ) have the same open and closed sets.

(d) Prove that if ρ and γ are equivalent metrics, then the following holds: If $\{x_n\}_n$ is a sequence in X, then $\{x_n\}_n$ converges in (X,ρ) if and only if it converges in (X,γ).

(e) Prove that if ρ and γ are equivalent metrics then (X,ρ) and (X,γ) have the same compact subsets.

(f) Let ρ be any metric on X, and define

$$\gamma(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)}. \tag{1}$$

(i) Prove that (X,γ) is bounded.

(ii) Prove that γ is a metric on X.

(iii) Prove that ρ and γ are equivalent.

Remark. Equivalent metrics share many properties, such as convergence of sequences (as shown in (d)) and continuity of functions (as can be shown in a similar way). It is sometimes more convenient to work with bounded metrics, in which case the trick in problem (f) can be used – given an unbounded metric ρ, define the bounded metric γ in (1), prove whatever properties you wish using γ, and then use equivalence to deduce the same properties for ρ.

Problem 3. Consider first the following definition:

Definition. Let (X,d) be a metric space and let $f : X \to \mathbb{R}$ be a bounded function (that is, there is some $M > 0$ such that $|f(x)| \leqslant M$ for all $x \in X$). Then the *lower and upper Moreau–Yosida* approximations* of f are the two functions $f_k^-, f_k^+ : X \to \mathbb{R}$ defined by

$$f_k^-(x) = \inf_{y\in X}\big(f(y) + kd(x,y)\big), \qquad f_k^+(x) = \sup_{y\in X}\big(f(y) - kd(x,y)\big)$$

where $k \in \mathbb{N}$ is a fixed number.

(a) Let $X = \mathbb{R}$, let $d(x,y) = |x-y|$, and fix some $k \in \mathbb{N}$. Compute and draw the graphs of f_k^- and f_k^+ for the functions†

$$f = \mathbb{1}_{(-\infty,0]} \qquad \text{and} \qquad f = \mathbb{1}_{[0,2]}.$$

For the rest of the exercise, let (X,d) be an arbitrary metric space and $f : X \to \mathbb{R}$ an arbitrary bounded function.

(b) Show that $f_k^-(x) \leqslant f(x) \leqslant f_k^+(x)$ for every $x \in X$.

(c) Show that if $a, b \in \mathbb{R}$ are such that $a \leqslant f(x) \leqslant b$ for every $x \in X$, then also $a \leqslant f_k^-(x) \leqslant b$ and $a \leqslant f_k^+(x) \leqslant b$ for every $x \in X$.

For the next problems we will concentrate on the lower Moreau–Yosida approximation f_k^-, although the same statements also hold for f_k^+.

(d) *(Only if you have the time!)* Show that f_k^- is Lipschitz continuous, with Lipschitz constant no larger than k.

(e) Show that if f is both continuous and bounded then $f_k^- \to f$ pointwise as $k \to \infty$.

(f) *(Only if you have the time!)* Show that if f is both uniformly continuous and bounded then $f_k^- \to f$ uniformly as $k \to \infty$.

Remark. It's not too hard to write a computer program that computes the Moreau–Yosida approximation for a function $f : \mathbb{R} \to \mathbb{R}$. A Matlab-like code to compute f_k^- would look like:

```
x = linspace(-2,4,1000)
fval = f(x)
fkMin = zeros(1,1000)
for i = 1..1000
    fkMin(i) = min(fval + k*abs(x(i)-x))
end
```

(here the domain is restricted to $x, y \in [-2,4]$). It would be a good idea to experiment with different f and k to get an idea of what f_k^- and f_k^+ look like.

*Pronounced *mårijå jåssida*

†If A is any set then the identity function $\mathbb{1}_A$ is defined as $\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$

Remark. A common technique in mathematical analysis is to approximate a (possibly very discontinuous) f by a "nicer" (say, continuous, Lipschitz continuous, differentiable, etc.) function f_k, prove some properties about f_k (which is easier to work with, since it is "nice"), and then let $k \to \infty$, hoping that $f = \lim_{k\to\infty} f_k$ inherits those properties. There are many ways of approximating a function with "nicer" functions, and the Moreau–Yosida approximation is one of them – it replaces f by a Lipschitz continuous function (cf. **3**(d)) which is close to f (cf. **3**(e),(f)) and satisfies some of the same properties as f (cf. **3**(b),(c)).

Hints

You might need the following facts:

1. The inverse triangle inequality: If (X,d) is a metric space then
$$\big|d(x,z)-d(z,y)\big| \leqslant d(x,y) \qquad \forall\ x,y,z \in X$$
2. If $g : X \to \mathbb{R}$ is a lower bounded function, then for every $\varepsilon > 0$ there is some $z \in X$ such that
$$\inf_{x\in X} g(x) \geqslant g(z) - \varepsilon.$$
3. If $g : X \to \mathbb{R}$ is any function then
$$-\inf_{x\in X} g(x) = \sup_{x\in X} \big(-g(x)\big).$$

Problem 1. Try to draw $B(0;1)$ first to get an idea of how each ball looks like.

Problem 2. (a) You may use – without proof[‡] – the inequalities
$$\|z\|_1 \leqslant \sqrt{2}\|z\|_2 \qquad \text{and} \qquad \|z\|_2 \leqslant \|z\|_1 \qquad \forall\ z \in \mathbb{R}^2$$
where $\|z\|_1 = |z_1| + |z_2|$ and $\|z\|_2 = \sqrt{(z_1)^2 + (z_2)^2}$.

(b) First figure out how the open balls in $(\mathbb{R}^2, d)$ look like.

(c) It helps to draw a sketch.

(f) (ii) For the triangle inequality, write $\gamma(x,y) = \varphi(\rho(x,y))$, where
$$\varphi(s) = \frac{s}{1+s}$$
Prove that φ is an increasing and *subadditive* function, that is, $\varphi(s+t) \leqslant \varphi(s) + \varphi(t)$ for all $s,t \geqslant 0$. (For the latter, write out $\varphi(s+t)$ and compare to $\varphi(s)+\varphi(t)$.)

Problem 3. (d) Your goal is to estimate $|f_k^-(x_1) - f_k^-(x_2)|$ in terms of (a constant times) $d(x_1,x_2)$. Now write out
$$f_k^-(x_1) - f_k^-(x_2) = \inf_{y_1\in X}\big(f(y_1) + kd(x_1,y_1)\big) - \inf_{y_2\in X}\big(f(y_2) + kd(x_2,y_2)\big).$$
Then use "fact number 2" above for the second term: There is some $z \in X$ such that $f_k^-(x_2) \geqslant f(z) + kd(x_2,z) - \varepsilon$. Insert into the expression above, and then use $y_1 = z$ in the first term.

(e) First use continuity of f: For $x \in X$ and $\varepsilon > 0$, let $\delta > 0$ be such that $|f(x) - f(y)| < \varepsilon$ when $d(x,y) < \delta$. Write $f_k^-(x) = \min(T_1(x),\ T_2(x))$, where
$$T_1(x) = \inf_{y\in B(x;\delta)}\big(f(y) + kd(x,y)\big), \qquad T_2(x) = \inf_{y\notin B(x;\delta)}\big(f(y) + kd(x,y)\big).$$
Then show that
$$T_1(x) \leqslant M \qquad \text{and} \qquad T_2(x) \geqslant -M + k\delta,$$
where $M > 0$ is such that $|f(x)| \leqslant M$ for all $x \in X$. If k is very large, which of T_1 and T_2 is smallest?

(f) You do not need to repeat the entire argument, just note that the choice of δ in the previous problem now does not depend on the choice of $x \in X$.

[‡]The first inequality can be proved by concavity of the function $s \mapsto \sqrt{s}$, and the second inequality is easily proved after squaring both sides.