MAT2400: Mandatory assignment #2, Spring 2020

Suggested solution

Problem 1.

(a) We compute first the Gateaux derivatives:

$$\lim_{h\to 0}\frac{F(A+hR)-F(A)}{h}=\lim_{h\to 0}\frac{A^2+hAR+hRA+hR^2-A^2}{h}=\lim_{h\to 0}\big(AR+RA+hR^2\big)=AR+RA$$

for any $A,R\in M$. This indicates that $F'(A)(R)=AR+RA$. Indeed,

$$F(A+R)-F(A)-(AR+RA)=A^2+AR+RA+R^2-A^2-AR-RA=R^2=o(\|R\|_{\mathcal{L}})$$

as $R\to 0$. It remains to prove that $F'(A)\in\mathcal{L}(M)$. It is clear that $F'(A)$ is linear, and moreover $\|F'(A)(R)\|_M=\|AR+RA\|_M\leqslant 2\|A\|_M\|R\|_M$, so $F'(A)$ is bounded with $\|F'(A)\|_{\mathcal{L}(M)}\leqslant 2\|A\|_M$.

(b) *There was a typo in the problem: It should have been I_Y, not I_M.*

We wish to apply the inverse function theorem. The function F is Fréchet differentiable, and its derivative is continuous since

$$\begin{aligned}\|F'(A)-F'(B)\|_{\mathcal{L}(M)}&=\sup_{\substack{R\in M\\ \|R\|_M=1}}\big\|(AR+RA)-(BR+RB)\big\|_M\\&\leqslant\sup_{\substack{R\in M\\ \|R\|_M=1}}\|(A-B)R\|_M+\sup_{\substack{R\in M\\ \|R\|_M=1}}\|R(A-B)\|_M\\&\leqslant\|A-B\|_M+\|B-A\|_M=2\|A-B\|_M\end{aligned}$$

(so F' is Lipschitz continuous with constant 2). Moreover, at $A=I_Y$ the derivative satisfies $F'(I_Y)(R)=2R$, so $F'(I_Y)=2I_M$, which is invertible (with inverse $F'(I_Y)=\frac{1}{2}I_M$). Hence, there is some $\varepsilon>0$ such that F is invertible in $B(I_Y;\varepsilon)$. If $A=F^{-1}(B)$ then $B=F(A)=A^2$. The operator A is not unique, since also $F(-A)=B$.

Problem 2.

(a) Let $x\in X$ and let $\{x_n\}_n$ be any sequence in D converging to x. Then the sequence $\{Ax_n\}_n$ is Cauchy in Y, since $\|Ax_n-Ax_m\|_Y\leqslant\|A\|_{\mathcal{L}}\|x_n-x_m\|_X$, and $\{x_n\}_n$ is Cauchy. Since Y is complete, there is some $y\in Y$ such that $Ax_n\to y$ as $n\to\infty$. The point y is unique, for if $\{x_n'\}_n$ is another sequence converging to x then $\|y-Ax_n'\|_Y\leqslant\|y-Ax_n\|_Y+\|Ax_n-Ax_n'\|_Y\leqslant\|y-Ax_n\|_Y+\|A\|_{\mathcal{L}}\|x_n-x_n'\|_X\to 0$ as $n\to\infty$.

(b) If $x\in D$ then $x_n=x$ (for all $n\in\mathbb{N}$) defines a sequence in D converging to x. Hence, $B(x)=\lim_{n\to\infty}Ax_n=Ax$.

(c) Let $\alpha\in\mathbb{K}$, let $x,y\in X$ and let $\{x_n\}_n$ and $\{y_n\}_n$ be sequences in D converging to x and y, respectively. Then $\alpha x_n+y_n\to\alpha x+y$, so

$$B(\alpha x+y)=\lim_{n\to\infty}A(\alpha x_n+y_n)=\lim_{n\to\infty}\big(\alpha A(x_n)+A(y_n)\big)=\alpha\lim_{n\to\infty}A(x_n)+\lim_{n\to\infty}A(y_n)=\alpha B(x)+B(y).$$

Hence, B is linear. To see that B is bounded, let $x\in X$ and let $\{x_n\}_n$ be a sequence in D converging to x. Then

$$\|B(x)\|_Y=\lim_{n\to\infty}\|A(x_n)\|_Y\leqslant\|A\|_{\mathcal{L}}\lim_{n\to\infty}\|x_n\|_X=\|A\|_{\mathcal{L}}\|x\|_X.$$

Thus, B is bounded with $\|B\|_{\mathcal{L}}\leqslant\|A\|_{\mathcal{L}}$. (In fact, $\|B\|_{\mathcal{L}}=\|A\|_{\mathcal{L}}$, since $B\big|_D=A$.)

(d) Let $\tilde{B}$ be another such operator. If $x\in X$ and $\{x_n\}_n$ is a sequence in D converging to x, then, by the triangle inequality,

$$\|\tilde{B}x-Bx\|_Y\leqslant\|\tilde{B}x-\tilde{B}x_n\|_Y+\|\tilde{B}x_n-Ax_n\|_Y+\|Ax_n-Bx\|_Y.$$

Here, $\|\tilde{B}x - \tilde{B}x_n\|_Y \to 0$ since $\tilde{B}$ is continuous; $\|\tilde{B}x_n - Ax_n\|_Y = 0$ since $\tilde{B}$ and A agree on D; and $\|Ax_n - Bx\|_Y \to 0$, by definition of B. It follows that $\|\tilde{B}x - Bx\|_Y = 0$, whence $\tilde{B}x = Bx$. Since x was arbitrary, we conclude $\tilde{B} = B$.

Problem 3. There were two issues with Problem 3 that I regrettably did not notice when I wrote the problem. The issue arises when adding two functions $u \in D_n$ and $v \in D_m$ – at first, it might not even seem like D is a vector space. For completeness I prove this here:

Proposition. *Let $n \in \mathbb{N}$. Then $D_n \subset D_{nm}$ for every $m \in \mathbb{N}$. In particular, if $u \in D_n$ has the representation $u = \sum_{i=0}^{n} a_i v_{n,i}$ then it can be written as*

$$u = \sum_{k=0}^{nm} \tilde{a}_k v_{nm,k}, \qquad \tilde{a}_k = \sum_{i=0}^{n} a_i v_{n,i}(k/(nm)).$$

Proof. For every $n, \in \mathbb{N}$, the space D_n is the space of all $u \in C([0,1],\mathbb{R})$ that are affine (i.e., a first order polynomial) on each interval $[\frac{i}{n}, \frac{i+1}{n}]$. Since nm is divisible by n, we can write

$$[\tfrac{i}{n}, \tfrac{i+1}{n}] = \bigcup_{j=im}^{(i+1)m-1} [\tfrac{j}{nm}, \tfrac{j+1}{nm}].$$

Hence, a function which is affine on every interval $[\frac{i}{n}, \frac{i+1}{n}]$ is also affine on every interval $[\frac{j}{nm}, \frac{j+1}{nm}]$. It follows that every function in D_n also lies in D_{nm}.

To arrive at the representation formula, we note first that if $u \in D_n$ then

$$u = \sum_{i=0}^{n} u(i/n) v_{n,i}.$$

Hence,

$$u = \sum_{k=0}^{nm} \tilde{a}_k v_{nm,k}, \qquad \tilde{a}_k = u(k/(nm)) = \sum_{i=0}^{n} a_i v_{n,i}(k/(nm)).$$

□

A corollary of the proposition is that $D = \cup_{n\in\mathbb{N}} D_n$ is a vector space.

Since the proposition implies that every function $u \in D$ will have multiple (in fact, infinitely many) different representations, it might not be clear that F is well-defined – that is, that the value $F(u)$ does not depend on how we choose to represent u. I prove next that F is indeed well-defined.

Proposition. *The function F defined in 3(c) is well-defined.*

Proof. Let $n, m \in \mathbb{N}$ and let $u \in D_n$. By the above proposition we know that u can be written as

$$u = \sum_{i=0}^{n} a_i v_{n,i} = \sum_{k=0}^{nm} \tilde{a}_k v_{nm,k}, \qquad \text{where } \tilde{a}_k = \sum_{i=0}^{n} a_i v_{n,i}(k/(nm)).$$

We claim that $F(u)$ is independent on which representation we choose. Indeed,

$$\begin{aligned}
F(u) &= F\left(\sum_{k=0}^{nm} \tilde{a}_k v_{nm,k}\right) \\
&= \frac{1}{2nm}(\tilde{a}_0 + \tilde{a}_{nm}) + \frac{1}{nm}\sum_{k=1}^{nm-1}\left(\sum_{i=0}^{n} a_i v_{n,i}(k/(nm))\right) \\
&= \frac{1}{2nm}(a_0 + a_n) + \frac{1}{nm}\sum_{i=0}^{n} a_i \sum_{k=1}^{nm-1} v_{n,i}(k/(nm)) \\
&= \frac{a_0}{nm}\left(\frac{1}{2} + \sum_{k=1}^{nm-1} v_{n,0}(k/(nm))\right) + \frac{a_n}{nm}\left(\frac{1}{2} + \sum_{k=1}^{nm-1} v_{n,n}(k/(nm))\right) + \frac{1}{nm}\sum_{i=1}^{n-1} a_i \underbrace{\sum_{k=1}^{nm-1} \frac{v_{n,i}(k/(nm))}{m}}_{=m} \\
&= \frac{a_0}{nm}\underbrace{\left(\frac{1}{2} + \sum_{k=1}^{m}(1-k/m)\right)}_{=m/2} + \frac{a_n}{nm}\underbrace{\left(\frac{1}{2} + \sum_{k=m(n-1)}^{nm-1}(k/m - n + 1)\right)}_{=m/2} + \frac{1}{n}\sum_{i=1}^{n-1} a_i \\
&= \frac{1}{2n}(a_0 + a_n) + \frac{1}{n}\sum_{i=1}^{n-1} a_i \\
&= F\left(\sum_{i=0}^{n} a_i v_{n,i}\right).
\end{aligned}$$

□

(a) Let $u \in D_n$. We claim that $\|u\| = \bar{u} := \max_{i=0,1,\dots,n}|a_i|$. Indeed, if $t \in [i/n, (i+1)/n]$ then $u(t) = a_i v_{n,i}(t) + a_{i+1} v_{n,i+1}(t) = a_i v_{n,i}(t) + a_{i+1}(1 - v_{n,i}(t))$, so

$$\begin{aligned}
|u(t)| &\leqslant |a_i| v_{n,i}(t) + |a_{i+1}|(1 - v_{n,i}(t)) \\
&\leqslant \bar{a} v_{n,i}(t) + \bar{a}(1 - v_{n,i}(t)) = \bar{a}.
\end{aligned}$$

Thus, $\|u\| \leqslant \bar{a}$. Conversely, if the maximum is attained at $|a_i| = \bar{a}$ then $|u(i/n)| = |a_i| = \bar{a}$, so $\|u\| \geqslant \bar{a}$.

(b) Let $u \in X$ and let $\varepsilon > 0$. Since $[0,1]$ is compact, u is uniformly continuous, so there is some $\delta > 0$ such that $|u(t) - u(s)| < \varepsilon$ whenever $|t - s| < \delta$. Let $n \in \mathbb{N}$ be such that $1/n < \delta$. Define $a_i = u(i/n)$ for $i = 0, 1, \dots, n$ and let $v(t) = \sum_{i=0}^{n} a_i v_{n,i}(t)$. Then $v \in D$, and if $t \in [0,1]$ lies in some interval $[i/n, (i+1)/n]$ then

$$\begin{aligned}
|u(t) - v(t)| &= \big|u(t) - u(i/n)v_{n,i}(t) - u((i+1)/n)v_{n,i+1}(t)\big| \\
&= \big|\big(u(t) - u(i/n)\big)v_{n,i}(t) + \big(u(t) - u((i+1)/n\big)v_{n,i+1}(t)\big| \\
&\leqslant \big|u(t) - u(i/n)\big|v_{n,i}(t) + \big|u(t) - u((i+1)/n\big|v_{n,i+1}(t) \\
&< \varepsilon v_{n,i}(t) + \varepsilon v_{n,i+1}(t) \\
&= \varepsilon.
\end{aligned}$$

Hence, $\|u - v\| = \max_{t\in[0,1]}|u(t) - v(t)| < \varepsilon$.

(c) F is clearly homogeneous: If $u(t) = \sum_{i=0}^{n} a_i v_{n,i}(t)$ then $\alpha u(t) = \sum_{i=0}^{n} b_i v_{n,i}(t)$, where $b_i = \alpha a_i$, so $\alpha u \in D$. To show that F is additive, let $u, v \in D$ and let $n, m \in \mathbb{N}$ be such that $u \in D_n$ and $v \in D_m$. By the first proposition above, we have $u, v \in D_{nm}$. Writing

$$u = \sum_{i=0}^{nm} a_i v_{nm,i}, \qquad v = \sum_{i=0}^{nm} b_i v_{nm,i},$$

we get

$$\begin{aligned}
F(u+v) &= \frac{1}{2nm}\left(a_0 + b_0 + a_{nm} + b_{nm}\right) + \frac{1}{nm}\sum_{i=0}^{nm} a_i + b_i \\
&= \left(\frac{1}{2nm}\left(a_0 + a_{nm}\right) + \frac{1}{nm}\sum_{i=0}^{nm} a_i\right) + \left(\frac{1}{2nm}\left(b_0 + b_{nm}\right) + \frac{1}{nm}\sum_{i=0}^{nm} b_i\right) \\
&= F(u) + F(v).
\end{aligned}$$

To see that u is bounded, let $u \in D_n$ and estimate

$$|F(u)| \leqslant \frac{1}{2n}(|a_0| + |a_1|) + \frac{1}{n}\sum_{i=1}^{n-1}|a_i| \leqslant \frac{1}{2n}(\|u\| + \|u\|) + \frac{1}{n}\sum_{i=1}^{n-1}\|u\| = \|u\|.$$

Thus, F is bounded with operator norm $\|F\|_{\mathcal{L}} \leqslant 1$. If $u \in D$ is constant, i.e. $a_0 = \cdots = a_n$ then $|F(u)| = \left|\frac{1}{2n}(a_0 + a_0) + \frac{1}{n}\sum_{i=1}^{n-1} a_0\right| = |a_0| = \|u\|$, so we conclude that $\|F\|_{\mathcal{L}} = 1$.

(d) This follows from Problem 1: F is a bounded linear functional from a dense subspace $D \subset X$ to the Banach space $Y = \mathbb{R}$.

(e) It is readily checked that $F(v_{n,i}) = \int_0^1 v_{n,i}(t)\,dt$, and hence $F(u) = \int_0^1 u(t)\,dt$ for all $u \in D$. If $u \in X$ and $\{v_n\}_n$ is a sequence in D then $G(u) = \lim_{n\to\infty} F(u_n) = \lim_{n\to\infty} \int_0^1 u_n(t)\,dt = \int_0^1 u(t)\,dt$, the last step following from Proposition 4.3.1 (continuity of the integral).