April 27, 2020

MAT2400 – Real Analysis

Mandatory assignment 2 of 2

Submission deadline

Thursday 30 April 2020, 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between writing in English or Norwegian. You will find a Norwegian–English mathematical dictionary at

uio.no/studier/emner/matnat/math/MAT2400/v20/norsk-engelsk-ordliste.html.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible; please use either the "Color" or "Grayscale" settings. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you **have to** contact the student administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Note: You will find a list of hints on page 3.

Problem 1. Let Y be a complete normed linear space and denote $M = \mathcal{L}(Y)$, the space of bounded linear operators $A : Y \to Y$. Let $F : M \to M$ be the map defined by

$$F(A) = A^2$$

(recall that A^2 denotes the composition $A^2 = A \circ A$).

(a) Show that F is Fréchet differentiable at each $A \in M$, and find the derivative.

(b) Show that there exists an $\varepsilon > 0$ such that every $B \in M$ with $\|B - I_M\|_{\mathcal{L}} < \varepsilon$ has a "square root" $\sqrt{B}$, i.e., there is a bounded linear operator A satisfying $A^2 = B$. Is such an A unique?

Problem 2. Let $\mathbb{K}$ be a field (either $\mathbb{R}$ or $\mathbb{C}$), let X, Y be normed vector spaces over $\mathbb{K}$, and assume that Y is a Banach space (i.e., a complete normed vector space). Let $D \subset X$ be a dense subspace and let $A \in \mathcal{L}(D, Y)$ be some bounded linear operator from D to Y. We will prove that there is a unique $B \in \mathcal{L}(X, Y)$ such that $Ax = Bx$ for every $x \in D$.

(a) For any $x \in X$, define $B(x) = y$, where $y = \lim_{n\to\infty} Ax_n$, and $\{x_n\}_n$ is any sequence in D converging to x. Prove that B is well-defined, that is, prove that $\{Ax_n\}_n$ indeed converges, and that the limit y is unique.

(b) Prove that the function B satisfies $B(x) = Ax$ for all $x \in D$.

(c) Prove that $B \in \mathcal{L}(X, Y)$.

(d) Prove that B is unique, in the sense that if $\tilde{B} \in \mathcal{L}(X, Y)$ also satisfies $\tilde{B}x = Ax$ for all $x \in D$, then $B = \tilde{B}$.

(The operator B is the *continuous linear extension* of A. The significance of this result is that in order to construct a bounded linear operator, it is enough to define it on a dense subspace. This technique is carried out in the next problem.)

Problem 3. In this problem we will construct the integral $\int_0^1 u(t)\,dt$ for any continuous function u, based on the integral of simpler, piecewise linear functions. Let $X = C([0,1], \mathbb{R})$ equipped with the supremum norm, and define the "hat functions"

$$v_{n,i}(t) = \begin{cases} nt - (i-1) & \text{if } (i-1)/n < t < i/n \\ i + 1 - nt & \text{if } i/n \leqslant t < (i+1)/n \\ 0 & \text{otherwise} \end{cases}$$

where $n \in \mathbb{N}$ and $i = 0, 1 \ldots, n$. Let $D_n \subset X$ be the subspace

$$D_n = \left\{ \sum_{i=0}^{n} a_i v_{n,i} \;\middle|\; a_0, a_1, \ldots, a_n \in \mathbb{R} \right\}$$

and let $D = \bigcup_{n\in\mathbb{N}} D_n$. (See Figure 1.)

(a) Prove that if $u \in D$ then $\|u\| = \max_{i=0,1,\ldots,n} |a_i|$.

(b) Prove that D is dense in X.

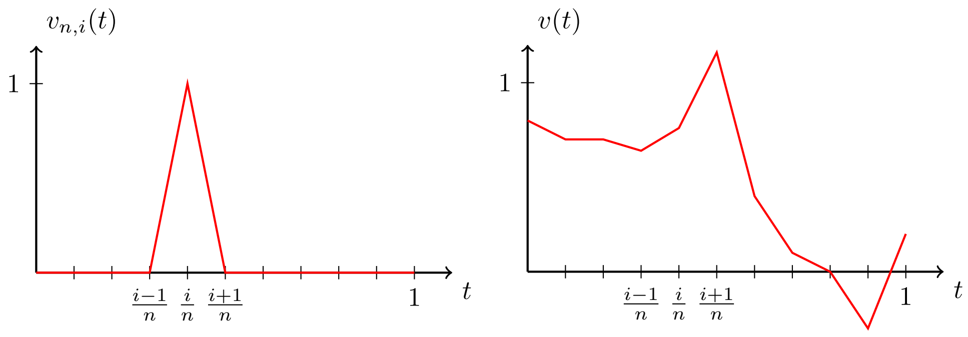

Figure 1: Left: One of the "hat functions" $v_{n,i}$. Right: A "typical" function in D_n.

(c) Let $F : D \to \mathbb{R}$ be the function

$$F(u) = \frac{1}{2n}(a_0 + a_n) + \frac{1}{n}\sum_{i=1}^{n-1} a_i$$

where $u(x) = \sum_{i=0}^{n} a_i v_{n,i}(x)$. Prove that F is a bounded linear functional on D (you do not need to show that F is additive) and show that $\|F\|_{\mathcal{L}} = 1$. (Here and elsewhere, we equip $\mathbb{R}$ with the canonical norm $\|y\| = |y|$.)

(d) Prove that there is a unique bounded linear functional $G : X \to \mathbb{R}$ such that $G(u) = F(u)$ for all $u \in D$.

(e) Prove that $G(u) = \int_0^1 u(t)\,dt$ for all $u \in X$.

Hints

Problem 2. (a) Show first that the sequence $\{y_n\}_n$ in Y defined by $y_n = Ax_n$ is Cauchy. Then use completeness of Y.

Problem 3. Let $t \in [0,1]$. Then there is some index $i \in \{0,1,\dots,n\}$ such that $t \in [i/n,(i+1)/n)$. Since $v_{n,j}(t) = 0$ for every j except $j = i$ and $j = i+1$, we get

$$v(t) = a_i v_{n,i}(t) + a_{i+1} v_{n,i+1}(t)$$

for any $v \in D_n$. Note also that in this interval we can write

$$v_{n,i+1}(t) = 1 - v_{n,i}(t).$$

Remember also that every continuous function on $[0,1]$ is uniformly continuous and attains a maximum/minimum at some point $t \in [0,1]$.

(a) One way of proving this is by first showing that if $u \in D$ attains its maximum/minimum at t, then $t = i/n$ for some i.

(b) You need to use the fact that every $u \in X$ is uniformly continuous. Define a function $v \in D_n$ such that $v(i/n) = u(i/n)$ for every $i = 0,1,\dots,n$, and let n be large enough that $u(t)$ and $v(t)$ are also close at all other points $t \in [0,1]$.

(d) Use Problem 2.