Solutions to section 3.3 (and how to write)

The first mandatory assignment is coming up, and you should therefore think about what a nice hand-in should look like! Below I have written solutions for section 3.3. Take a look at the structure of the arguments, and try to answer the problems in the assignment in a similar fashion.[1] (There is, of course, nothing special about these particular solutions. You can get the exact same input from other sources, e.g. by reading the proofs in *Spaces*.)

1 First: An example

Take a look in *Spaces* by Tom Lindstrøm. All of the proofs and arguments are almost all text! We write 'formal mathematical arguments' in our usual 'informal language' to make them readable and easy to understand. Here's an example that illustrates this point.

Proposition 1. *Let* $a, b \geq 0$. *Then* $a + b \leq \sqrt{2}\sqrt{a^2 + b^2}$.

Ugly proof. $0 \leq (a-b)^2 = a^2 - 2ab + b^2 \Rightarrow 2ab \leq a^2 + b^2$. So $(a+b)^2 = a^2 + 2ab + b^2 \leq 2(a^2 + b^2)$. Then $a + b = \sqrt{(a+b)^2} \leq \sqrt{2(a^2+b^2)} = \sqrt{2}\sqrt{a^2+b^2}$. □

Nicer proof. Assume $a, b \geq 0$. Then $a^2 - 2ab + b^2 = (a-b)^2 \geq 0$, which is equivalent to $2ab \leq a^2 + b^2$. It follows that

$$(a+b)^2 = a^2 + 2ab + b^2 \leq 2(a^2 + b^2).$$

Thus, taking the square root of the expressions on both sides of the inequality, we obtain that

$$a + b \leq \sqrt{2}\sqrt{a^2 + b^2}$$

because the square root function preserves the order on nonnegative numbers. □

In my view it is much easier to understand the second proof, simply because you can read the second proof as an 'ordinary text'.[2] Try to do write like this yourself!

[1] You don't have to follow the exact set up I use here (e.g. gray background for solutions etc.), and you can also submit hand written solutions. The main takeaway from these notes should just be the content within the gray boxes. In particular how the proofs are structured.

[2] The second proof also illustrates a nice trick: Don't hesitate to put bigger expressions on a new line, to make the arguements easier to read!

2 Solutions

Exercise 3.3.1 (See the exercise text in the book).

Solution. ***a)*** Let $x \in X$ and $r > 0$. Recall that the open ball centered at x with radius r is the set

$$B(x, r) = \{y \in X \, : \, d(x, y) < r\}.$$

Assume first that $r \leq 1$. Then, if $y \in B(x, r)$, we know that $d(x, y) < 1$. Because d is the discrete metric on X this implies that $d(x, y) = 0$ or equivalently that $x = y$. Hence, we see that $B(x, r) \subseteq \{x\}$. As we clearly have $x \in B(x, r)$ we conclude that $B(x, r) = \{x\}$.

Now, suppose $r > 1$. If $y \in X$, then $d(x, y) \in \{0, 1\}$, so in particular $d(x, y) < r$. This shows that $y \in B(x, r)$, which means that $X \subseteq B(x, r)$. Clearly, $B(x, r) \subseteq X$ as well. We conclude that $B(x, r) = X$.

b) Let $A \subseteq X$. We will show that A is open. To do this it suffices to show that for any $a \in A$ there is $r > 0$ such that $B(a, r) \subseteq A$. In the proof of *a)* we saw that $B(a, 1) = \{a\} \subset A$, so we are already done! Because A was arbitrary, we conclude that all subsets of X are open.

A set is closed if and only if its complement is open. Hence, because all subsets of X are open, all subsets of X must also be closed! Indeed, if $A \subseteq X$ then A^c is open, which means that A is closed.

c) Assume that $f : X \to Y$ is a function, and let $V \subseteq Y$ be an open set. Then, by *b)*, we know that $f^{-1}(V)$ is open as well. Thus, proposition 3.3.10 tells us that f is continuous.

Exercise 3.3.2 Draw a ball in the Manhattan metric. Show that the Manhattan metric and the usual metric on $\mathbb{R}^2$ have the same open sets.

Solution. For $x, y \in \mathbb{R}^2$ we use the notation $x = (x_1, x_2), y = (y_1, y_2)$. Then the Manhattan metric is given by

$$d_M(x, y) = |x_1 - y_1| + |x_2 - y_2|.$$

Fix $r > 0$. It is not hard to see that $d_M(x, 0) = |x_1| + |x_2| = r$ determines a square in $\mathbb{R}^2$ with corners $(r, 0), (0, r), (-r, 0), (0, -r)$. Hence, the ball $B_M(0, r)$ is the inside of this square. A ball $B_M(y, r)$, $y \in \mathbb{R}^2$ will have the same shape, but be located with center in y instead. (You should of course actually try to draw this.)

Now we prove the second claim.[a] Note that for all $x, y \in \mathbb{R}^2$ we have[b]

$$\|x - y\| \leq d_M(x, y) \leq \sqrt{2}\|x - y\|.$$

Let $x \in \mathbb{R}^2$ and $r > 0$. Write $B_M(x, r)$ and $B_{\|\cdot\|}(x, r)$ for the corresponding balls in the Manhattan metric and 'usual metric' respectivly. From the inequalites above we get

$$B_{\|\cdot\|}(x,r) \subset B_M(x,\sqrt{2}r) \quad \text{and} \quad B_M(x,r) \subset B_{\|\cdot\|}(x,r).$$

This implies that the metrics have the same open sets! To see this, assume that $U \subseteq \mathbb{R}^2$ is an open set in the Manhattan metric d_M, and take $x \in U$. Because U is open we can find $r > 0$ such that $B_M(x,r) \subset U$. Then, from the above,

$$B_{\|\cdot\|}(x,r/\sqrt{2}) \subset B_M(x,r) \subset U.$$

This means exactly that U is open with respect to the usual metric. Hence, an open set w.r.t d_M is an open set w.r.t $d(x,y) = \|x-y\|$. The other direction is similar.

[a]The intuition is that you can fit a square inside a circle if the diagonal is shorter than the radius, and similary that you can fit a circle inside a square if the radius is smaller than $1/\sqrt{2}$ times the diagonal.

[b]The second inequality is actually what we proved on the first page of these notes. I leave the first inequality you.

Exercise 3.3.3. F is a nonempty closed subset of $\mathbb{R}$ with the usual metric. Show that $\sup F \in F$ and $\inf F \in F$. Find a bounded but not closed set such that $\sup F \in F$ and $\inf F \in F$.

Solution. Note that for each $n \in \mathbb{N}$ we can find $x_n \in F$ such that

$$\sup F - \frac{1}{n} \leq x_n \leq \sup F.$$

Indeed, if we couldn't do this there would be m such that

$$x \leq \sup F - \frac{1}{m} \leq \sup F$$

for all $x \in F$, and this contradicts the definition of $\sup F$. Clearly $x_n \to \sup F$ (because of the 'squeeze law') and because F is closed we conclude that $\sup F \in F$. A similar argument shows that $\inf F \in F$.

The easiest example of a bounded but not closed set F such that $\sup F \in F$ is probably $F = (0,1]$. Then $\sup F = 1 \in F$, but F is not closed because $0 \notin F$.

Exercise 3.3.7. Let A be a subset of a metric space (X,d). Show that the interior points of A are the exterior points of A^c, and that the exterior points of A are the interior points of A^c. Check that the boundary points of A are the boundary points of A^c.

Solution. Let $x \in X$. Recall that x is called

- an *interior point* of A if there is $r > 0$ such that $B(x,r) \subset A$,
- an *exterior point* of A is there is $r > 0$ such that $B(x,r) \subset A^c$.

- a *boundary point* if of A all $r > 0$

$$B(x,r) \cap A \neq \emptyset \quad \text{and} \quad B(x,r) \cap A^c \neq \emptyset.$$

If you apply the first definition to A^c, you see that the definition of an interior point of A^c is exactly the same as the definition of an exterioir point of A. Similarily, applying the second definition to A^c we see (by the first definition) that an exterior point of A^c is an interior point of $(A^c)^c = A$.

Using the third definition on A^c, 'nothing happens' because $A = (A^c)^c$. We conclude that A and A^c have the same boundary points.

Exercise 3.3.11 Prove proposition 3.3.12. Find an example of an infinite collection of open sets whose intersection is *not* open.

3.3.12 *a)* Let $\mathcal{G}$ be a (possibly infinite) collection of open sets. Take $x \in \bigcup_{G\in\mathcal{G}} G$. By definition of union, there is $G' \in \mathcal{G}$ such that $x \in G'$. Because G' is open, there is $r > 0$ such that $B(x,r) \subset G'$. But then

$$B(x,r) \subset G' \subset \bigcup_{G\in\mathcal{G}} G.$$

This shows that $\bigcup_{G\in\mathcal{G}} G$ is open.

3.3.12 *b)* Assume now that $\mathcal{G} = \{G_1, G_2, ..., G_n\}$ is a *finite* collection of open sets. Take $x \in \bigcap_{G\in\mathcal{G}} G = G_1 \cap G_2 \cap \cdots \cap G_n$. Then $x \in G_i$ for all $i = 1, 2, ..., n$. As G_i is open, we can find $r_i > 0$ such that $B(x, r_i) \subset G_i$. Put $r = \min\{r_1, r_2, ..., r_n\}$. Then

$$B(x,r) \subset B(x,r_1) \cap B(x,r_2) \cap \cdots B(x,r_n) \subset \bigcap_{G\in\mathcal{G}} G,$$

showing that $\bigcap_{G\in\mathcal{G}} G$ is open.

Example: For each $n \in \mathbb{N}$ consider the open interval $I_n = (-1/n, 1/n) \subset \mathbb{R}$, and let $\mathcal{G} = \{I_n\}_n$ be the family consisting of all these intervals. Then

$$\bigcap_{G\in\mathcal{G}} G = \bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \{0\}.$$

To see this note that $0 \in I_n$ for all $n \in \mathbb{N}$, and that in fact this is the only number lying in all of the intervals (why?).

Ulriks's exerxise 1. Let $f : X \to Y$ be a continuous function between metric spaces. Fix $y \in Y$. Show that the set of solutions to the equation $f(x) = y$ is closed.

Solution. The set in question is

$$\{x \in X \,:\, f(x) = y\} = \big\{x \in X \,:\, f(x) \in \{y\}\big\} = f^{-1}(\{y\}).$$

$\{y\}$ is a closed set (see exercise last week), and hence $f^{-1}(\{x\})$ is also closed because f is continuous (Prop. 3.3.11).

Ulrik's exercise 2. Show that the support (as defined by Ulrik) of a continuous function on $\mathbb{R}$ (with the usual metric) is open .

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. The support $\mathsf{supp} f$ of f is the set of points $x \in \mathbb{R}$ such that $f(x) \neq 0$. That is

$$\begin{aligned}\mathsf{supp} f &= \{x \in \mathbb{R} \,:\, f(x) \neq 0\} \\ &= \big\{x \in \mathbb{R} \,:\, f(x) \in \{0\}^c\big\} = f^{-1}\big(\{0\}^c\big).\end{aligned}$$

As $\{0\}^c$ is an open set and f is continuous we conclude that $\mathsf{supp} f$ is open (see prop. 3.3.10).