Updated March 25, 2021 (with changes in red)

MAT2400 – Real Analysis

Mandatory assignment 1 of 2

Submission deadline

Thursday 25 February 2021, 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between writing in English or Norwegian. You will find a Norwegian–English mathematical dictionary at

https://www.uio.no/studier/emner/matnat/math/MAT2400/data/norsk-engelsk-ordliste.html.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible; please use either the "Color" or "Grayscale" settings. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Note: You will find a list of hints on page 8.

Problem 1.

(a) Show that we can always interchange two sup's: If A and B are sets and $f\colon A \times B \to \mathbb{R}$ is a function, then

$$\sup_{x\in A}\sup_{y\in B} f(x,y) = \sup_{y\in B}\sup_{x\in A} f(x,y) \tag{1}$$

To avoid dealing with infinities, you can assume that f is bounded from above.

Note: The same holds for two inf*'s.*

(b) We can *not* always interchange an inf and a sup: Show that if $f(x,y) = (x+y)^2$ then

$$\sup_{x\in A}\inf_{y\in B} f(x,y) \neq \inf_{y\in B}\sup_{x\in A} f(x,y)$$

where $A = B = [-1,1]$.

Recall that $\sup_{x\in A} g(x)$ *(for some function* g*) means the same as* $\sup\{g(x) : x \in A\}$.

Solution:

(a) Let $\alpha = \sup_{(x,y)\in A\times B} f(x,y)$. We claim that both sides of (1) equal α. First, $f(x,y) \leqslant \alpha$ for any (x,y), so $\sup_{y\in B} f(x,y) \leqslant \alpha$ for any $x \in A$, so it follows that $\sup_{x\in A}\sup_{y\in B} f(x,y) \leqslant \alpha$. For the converse, note that

$$\sup_{y\in B} f(a,y) \geqslant f(a,b)$$

for any fixed $a \in A$ and $b \in B$. Hence, taking the supremum over a on the left-hand side yields

$$\sup_{x\in A}\sup_{y\in B} f(x,y) \geqslant f(a,b)$$

for any fixed $a \in A$ and $b \in B$. In other words, the left-hand side is an upper bound for f, so

$$\sup_{x\in A}\sup_{y\in B} f(x,y) \geqslant \alpha.$$

We conclude that α equals the left-hand side of (1). By symmetry, the same goes for the right-hand side.

Alternatively: For every $(a,b) \in A \times B$, we have

$$f(a,b) \leqslant \sup_{y\in B} f(a,y) \leqslant \sup_{x\in A}\sup_{y\in B} f(x,y).$$

Taking the supremum over $a \in A$ and then over $b \in B$ gives

$$\sup_{b\in B}\sup_{a\in A} f(a,b) \leqslant \sup_{x\in A}\sup_{y\in B} f(x,y).$$

Arguing in the opposite order gives "$\geqslant$", and hence, there is equality in the above.

(b) For any $x \in A$ we have

$$\inf_{y\in B} f(x,y) = \inf_{y\in[-1,1]} (x+y)^2 = (x-(-x))^2 = 0$$

so $\sup_{x\in A} \inf_{y\in B} f(x,y) = 0$. On the other hand,

$$\sup_{x\in A} f(x,y) = \sup_{x\in[-1,1]} (x+y)^2 = \begin{cases} (1+y)^2 & \text{if } y \geqslant 0 \\ ((-1)+y)^2 & \text{if } y < 0, \end{cases}$$

so $\inf_{y\in B} \sup_{x\in A} f(x,y) = (1-0)^2 = 1$.

Problem 2. In this exercise we will investigate the limit of "double sequences" – sets of objects $x_{i,j}$ in some metric space (X,d) indexed over $i,j \in \mathbb{N}$. The question is whether we can interchange limits in i and j:

$$\lim_{i\to\infty} \lim_{j\to\infty} x_{i,j} = \lim_{j\to\infty} \lim_{i\to\infty} x_{i,j}. \tag{2}$$

See Figure 1 for an illustration. Note that we need to assume that all of the limits

$$y_i = \lim_{j\to\infty} x_{i,j}, \quad y = \lim_{i\to\infty} y_i, \quad z_j = \lim_{i\to\infty} x_{i,j}, \quad z = \lim_{j\to\infty} z_j \tag{3}$$

exist – otherwise (2) would not make sense.

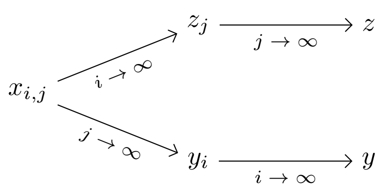

Figure 1: Is $y = z$?

(a) Consider the metric space $\mathbb{R}$ with the canonical metric, and let $x_{i,j} = i\min(1/i, 1/j)$. Show that (2) is *not* true.

(b) We say that the limit $\lim_{j\to\infty} x_{i,j}$ *is uniform in i* if for every $\varepsilon > 0$ there is some $M \in \mathbb{N}$ such that

$$d(x_{i,j}, y_i) < \varepsilon \quad \text{for all } j \geqslant M \text{ and all } i \in \mathbb{N}.$$

For the example in problem **(a)**, show that the limit is *not* uniform in i.

(c) Show that there is always some subsequence $\{x_{I(j),j}\}_{j\in\mathbb{N}}$ converging to z, and another subsequence $\{x_{i,J(i)}\}_{i\in\mathbb{N}}$ converging to y. (Here, both of $I, J\colon \mathbb{N} \to \mathbb{N}$ are increasing sequences, as in the definition of a subsequence.)

(d) Show that if the limit $\lim_{j\to\infty} x_{i,j}$ is uniform in i, then (2) is true.

Remark. The morale is: If you want to interchange two limits, then one of them has to be uniform.

Solution:

(a) We can write

$$x_{i,j} = \begin{cases} 1 & \text{if } i \geqslant j \\ \frac{i}{j} & \text{if } i < j. \end{cases}$$

Thus, $y_i = \lim_j x_{i,j} = 0$, while $z_j = \lim_i x_{i,j} = 1$. Clearly, $y_i \to 0$ but $z_j \to 1$ as $i, j \to \infty$.

(b) If $\varepsilon < 1$ then, no matter the choice of j, we can let $i = j$ to get

$$d(x_{i,j}, y_i) = d(x_{j,j}, 0) = x_{j,j} = 1 \not< \varepsilon.$$

Hence, the limit is not uniform in i.

(c) For each $j \in \mathbb{N}$, let $I(j)$ be such that $d(x_{i,j}, z_j) < 1/j$ whenever $i \geqslant I(j)$. (Such a number $I(j)$ exists because $x_{i,j} \to z_j$ as $i \to \infty$ for every fixed j.) Then

$$d(x_{I(j),j}, z) \leqslant d(x_{I(j),j}, z_j) + d(z_j, z) < 1/j + d(z_j, z) \to 0$$

as $j \to \infty$. The same idea works for the other limit.

(d) Let $\varepsilon > 0$ and let $M \in \mathbb{N}$ be such that

$$d(x_{i,j}, y_i) < \varepsilon \qquad \text{when } j \geqslant M, \text{ for any } i \in \mathbb{N}.$$

Passing $i \to \infty$ in this inequality gives $d(z_j, y) \leqslant \varepsilon$. Finally, passing $j \to \infty$ gives $d(z, y) \leqslant \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we must have $d(z, y) = 0$, whence $z = y$.

Alternative solution: Let $\varepsilon > 0$ and let $M \in \mathbb{N}$ be such that $d(x_{i,j}, y_i) < \varepsilon$ when $j \geqslant M$. Next, let $\widetilde{M}$ be such that $d(z_j, z) < \varepsilon$ when $j \geqslant \widetilde{M}$. Set $J = \max(M, \widetilde{M})$. Finally, let N be such that $d(x_{i,J}, z_J) < \varepsilon$ when $i \geqslant N$. Then

$$d(y_i, z) \leqslant d(y_i, x_{i,J}) + d(x_{i,J}, z_J) + d(z_J, z) < 3\varepsilon$$

for all $i \geqslant N$. Thus, $y_i \to z$ as $i \to \infty$, that is,

$$y_i = \lim_{j\to\infty} x_{i,j} \to z = \lim_{j\to\infty} \lim_{i\to\infty} x_{i,j}$$

as $i \to \infty$. This completes the proof.

Problem 3. Let (X, d) be a metric space. A function $f\colon X \to \mathbb{R}$ is *lower semicontinuous* at $x \in X$ if for every $\varepsilon > 0$, there is some $\delta > 0$ such that

$$f(x) < f(y) + \varepsilon \qquad \forall\, y \in B(x; \delta). \tag{4}$$

It is *upper semicontinuous* at $x \in X$ if for every $\varepsilon > 0$, there is some $\delta > 0$ such that

$$f(x) > f(y) - \varepsilon \qquad \forall\, y \in B(x;\delta). \tag{5}$$

(a) Show that a function is continuous at x *if and only if* it is both upper and lower semicontinuous at x.

(b) Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x & x < 0 \\ x+1 & x \geqslant 0 \end{cases}$$

is upper semicontinuous, but not lower semicontinuous.

(c) Show that if $K \subset X$ is compact and $f\colon K \to \mathbb{R}$ is lower semicontinuous then f attains a minimum – there is some $\bar{x} \in K$ such that

$$f(\bar{x}) \leqslant f(x) \qquad \forall\, x \in K. \tag{6}$$

Remark. Lower/upper semicontinuous functions are encountered in many places, particularly in the field of calculus of variations, where one attempts to find minima or maxima of functions. As seen in problems **(a)** and **(b)**, upper/lower semicontinuity is a *weaker* property than continuity, but, as seen in problem **(c)**, semicontinuous functions nonetheless have some of the important properties that continuous functions have.

Solution:

(a) If x is continuous then, for every $\varepsilon > 0$, there is some $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \qquad \forall\, y \in B(x;\delta),$$

which is the same as

$$-\varepsilon < f(x) - f(y) < \varepsilon \qquad \forall\, y \in B(x;\delta),$$

which, again, is the same as

$$f(y) - \varepsilon < f(x) < f(y) + \varepsilon \qquad \forall\, y \in B(x;\delta),$$

so f is lower and upper semicontinuous at x.

If f is both lower and upper semicontinuous at x, then for every $\varepsilon > 0$, there are $\delta_1 > 0$ and $\delta_2 > 0$ such that (4) and (5) hold with $\delta = \delta_1$ and $\delta = \delta_2$, respectively. Hence, they both also hold for $\delta = \min(\delta_1, \delta_2)$:

$$f(y) - \varepsilon < f(x) < f(y) + \varepsilon \qquad \forall\, y \in B(x;\delta),$$

which, as we found out earlier, is the same as

$$|f(x) - f(y)| < \varepsilon \qquad \forall\, y \in B(x;\delta).$$

Thus, f is continuous at x.

(b) If $x \neq 0$ then f is continuous at x, so it's also semicontinuous there. If $x = 0$ then $f(x) = 1$, and if $y < x$ then $f(y) = y < f(x)$, while if $y \geqslant 0$ then $f(y) = 1 + y < f(x) + \varepsilon$ provided $y < \varepsilon$. Thus, (5) holds if $\delta = \varepsilon$, so f is upper semicontinuous.

On the other hand, f is not continuous at $x = 0$, so by 2(a), it cannot be both upper and lower semicontinuous there; since we have shown that it is upper semicontinuous, it cannot possibly also be lower semicontinuous.

Alternative solution: f is not lower semicontinuous at $x = 0$, since if $\varepsilon < 1$, $\delta > 0$ and $y \in (-\delta, 0)$ then $f(y) + \varepsilon = y + \varepsilon < \varepsilon < 1 = f(x)$, so (4) does *not* hold, no matter the value of δ.

(c) Let $m = \inf f(K)$. We claim that there is some $\bar{x} \in K$ such that $f(\bar{x}) = m$, at which point we're done.

Let $\{y_n\}_{n\in\mathbb{N}}$ be a minimizing sequence for $\inf f(K)$ — a sequence in $f(K)$ converging to m. For every $n \in \mathbb{N}$, let $x_n \in K$ be such that $f(x_n) = y_n$. Then $\{x_n\}_n$ is a sequence in the compact set K, so it has a convergent subsequence, $x_{n(k)} \to \bar{x} \in K$ as $k \to \infty$ for some $\bar{x} \in K$.

Let $\varepsilon > 0$ and let $\delta > 0$ be as in (4). Then, for N large enough that $x_{n(k)} \in B(\bar{x}; \delta)$ for every $k \geqslant N$, we get

$$f(\bar{x}) < f(x_{n(k)}) + \varepsilon \to m + \varepsilon \qquad \text{as } k \to \infty,$$

so $f(\bar{x}) < m + \varepsilon$ for every $\varepsilon > 0$. Since ε was arbitrary, we must have $f(\bar{x}) \leqslant m$. In particular, $\inf f(K) = m \geqslant f(\bar{x}) > -\infty$, so we find that f is bounded from below. But $f(\bar{x}) \geqslant \inf f(K) = m$, so $f(\bar{x}) = m$, as we wanted.

Problem 4. In this problem we will identify some of the compact subsets of the metric space

$$\ell^\infty(\mathbb{R}) = \big\{a \colon \mathbb{N} \to \mathbb{R} \ : \ \|a\|_{\ell^\infty} = \sup_{i\in\mathbb{N}} |a(i)| < \infty\big\},$$

equipped with the metric

$$d_\infty(a, b) = \|a - b\|_{\ell^\infty} = \sup_{i\in\mathbb{N}} |a(i) - b(i)|.$$

Remark. We can think of the elements of ℓ^∞ either as sequences $\{a_i\}_{i\in\mathbb{N}}$ (for $a_i \in \mathbb{R}$), as functions $a \colon \mathbb{N} \to \mathbb{R}$, or as infinite-dimensional vectors $a = (a_1, a_2, \dots)$.

(a) Show that the closed, bounded set $\overline{B}(0; 1)$ is not compact. For instance, you can let $x_n = (0, \dots, 0, 1, 0, \dots)$ (where the "1" appears in the nth position) and show that x_n lies in $\overline{B}(0; 1)$ for any $n \in \mathbb{N}$, but that $\{x_n\}_n$ does not have a convergent subsequence.

(b) Prove the following:

Theorem. *Let $K \subset \ell^\infty(\mathbb{R})$ be any closed subset satisfying the following: There exists some $C > 0$ and $N \in \mathbb{N}$ such that every $a \in K$ satisfies*

$$|a(i)| \leqslant C \quad \forall\, i \leqslant N, \qquad a(i) = 0 \quad \forall\, i > N.$$

Then K is compact.

You are allowed to use the Bolzano–Weierstrass theorem: If $A \subset \mathbb{R}$ is closed and bounded then it's compact.

Remark. What the above result illustrates is that when we are dealing with infinite-dimensional spaces, we need some additional restriction on the "behaviour at infinity" in order to obtain compactness.

Solution:

(a) We have $d_\infty(x_n, x_m) = 1$ when $n \neq m$. If there were a convergent subsequence $\{x_{n(k)}\}_k$ then

$$1 = d_\infty(x_{n(k)}, x_{n(l)}) \to 0 \qquad \text{as } k, l \to \infty,$$

a contradiction.

(b) Let $\{a_n\}_n$ be a sequence in K. Then $\{a_n(1)\}_n$ is a bounded sequence in $\mathbb{R}$, so by Bolzano–Weierstrass it has a subsequence $\{a_{n_1(k)}(1)\}_{k\in\mathbb{N}}$ converging to some $a(1) \in \mathbb{R}$. Likewise, $\{a_{n_1(k)}(2)\}_{k\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$, so by Bolzano–Weierstrass it has a subsequence $\{a_{n_2(k)}(2)\}_{k\in\mathbb{N}}$ converging to some $a(2) \in \mathbb{R}$. Note that since $\{a_{n_2(k)}(1)\}_{k\in\mathbb{N}}$ is a subsequence of $\{a_{n_1(k)}(1)\}_{k\in\mathbb{N}}$, we still have $a_{n_2(k)}(1) \to a(1)$ as $k \to \infty$.

Iterating in this way N times, we end up with a sequence of natural numbers $n_N(k) \in \mathbb{N}$ (for $k \in \mathbb{N}$) such that $|a_{n_N(k)}(i) - a(i)| \to 0$ as $k \to \infty$ for every $i = 1, 2, \dots, N$. Thus, also $\max_{i=1,\dots,N} |a_{n_N(k)}(i) - a(i)| \to 0$ as $k \to \infty$.

Define now $a = \big(a(1), a(2), \dots, a(N), 0, 0, \dots\big) \in K$. Then

$$d_\infty(a_n, a) = \sup_{i\in\mathbb{N}} |a_n(i) - a(i)| = \max_{i=1,\dots,N} |a_n(i) - a(i)|$$

for every $n \in \mathbb{N}$. Hence, $d_\infty(a_{n_N(k)}, a) \to 0$ as $k \to \infty$, as we wanted.

Alternatively: Define $\Phi : \mathbb{R}^N \to \ell^\infty(\mathbb{R})$ by

$$\Phi(a) = (a(1), \dots, a(N), 0, 0, \dots) \qquad \forall\, a \in \mathbb{R}^N.$$

Then Φ is clearly injective. Let $B = \Phi^{-1}(K)$. By the assumption that $a(i) = 0$ for all $i > N$ for every $a \in K$, we see that $\Phi : B \to K$ is also surjective. If we equip $\mathbb{R}^N$ with the norm $\|a\|_\infty = \max(|a_1|, \dots, |a_N|)$ then we see that $d_{\ell^\infty}(\Phi(a), \Phi(b)) = \|a - b\|_\infty$ for all $a, b \in \mathbb{R}^N$ and $\|\Phi^{-1}(a) - \Phi^{-1}(b)\|_\infty = d_{\ell^\infty}(a, b)$ for all $a, b \in \ell^\infty(\mathbb{R})$, so both Φ and Φ^{-1} are continuous.

We see that B is the inverse image of a closed set under the continuous function Φ, hence B is closed. It is also bounded, since $\sup_{a\in B} \|a\|_\infty = \sup_{a\in B} d_{\ell^\infty}(\Phi(a), 0) = \sup_{a\in K} d_{\ell^\infty}(a, 0) < \infty$.

We have shown that $B \subset \mathbb{R}^N$ is closed and bounded, so by Bolzano–Weierstrass in $\mathbb{R}^N$, it is compact. But then $K = \Phi(B)$ is the forward image of a compact set under a continuous function, hence is compact.

Hints

Warning: Don't use these hints blindly! When writing your solution, *do not assume that the person who will correct your assignment has read the hints.*

Problem 2(a). Start by identifying the limits y_i and z_j.

Problem 2(d). Show that $y_i \to z$ as $i \to \infty$.

Problem 3(c). Can you modify the proof of the extreme value theorem to fit this setting?

Problem 4(b). Recall the proof of the fact that the compact subsets of $\mathbb{R}^N$ are those that are closed and bounded.