April 27, 2021

MAT2400 – Real Analysis

Mandatory assignment 2 of 2

Submission deadline

Monday 26 April 2021 at 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between writing in English or Norwegian. You will find a Norwegian–English mathematical dictionary at

https://www.uio.no/studier/emner/matnat/math/MAT2400/data/norsk-engelsk-ordliste.html.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible; please use either the "Color" or "Grayscale" settings. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Note: You will find a list of hints on page 7.

Problem 1. Let $f \in C([0,1],\mathbb{R})$. Use Weierstrass' approximation theorem to prove that

$$\lim_{n\to\infty} \int_0^1 f(t)\cos(nt)\,dt = 0. \tag{1}$$

Solution: By Weierstrass' approximation theorem, there is for every $\varepsilon > 0$ some polynomial p such that $\|f - p\|_\infty < \varepsilon$. We have

$$\begin{aligned}\left|\int_0^1 p(t)\cos(nt)\,dt\right| &= \left|\frac{p(1)\sin(n) - p(0)\sin(0)}{n} - \int_0^1 \frac{p'(t)\sin(nt)}{n}\,dt\right| \\ &\leqslant \frac{|p(1)|}{n} + \frac{\int_0^1 |p'(t)|\,dt}{n} \\ &\to 0\end{aligned}$$

as $n \to \infty$. Thus,

$$\begin{aligned}\left|\int_0^1 f(t)\cos(nt)\,dt\right| &\leqslant \left|\int_0^1 p(t)\cos(nt)\,dt\right| + \left|\int_0^1 \big(f(t) - p(t)\big)\cos(nt)\,dt\right| \\ &\leqslant \left|\int_0^1 p(t)\cos(nt)\,dt\right| + \int_0^1 \big|f(t) - p(t)\big|\,|\cos(nt)|\,dt \\ &\leqslant \left|\int_0^1 p(t)\cos(nt)\,dt\right| + \|f - p\|_\infty \\ &< \left|\int_0^1 p(t)\cos(nt)\,dt\right| + \varepsilon \\ &\to \varepsilon\end{aligned}$$

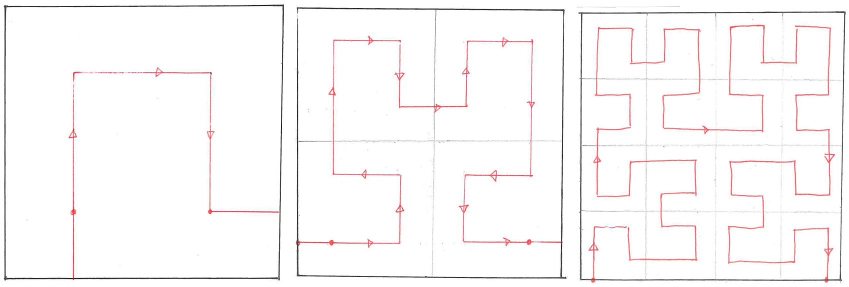
as $n \to \infty$. Since $\varepsilon > 0$ was arbitrary, we must have $\int_0^1 f(t)\cos(nt)\,dt \to 0$ as $n \to \infty$.

Problem 2. A *space-filling curve* is a continuous and surjective function $f\colon [0,1] \to [0,1]^2$, that is, a curve which fills the entire unit square. In this problem we will construct one such curve, the Hilbert curve.

Let $f_1 : [0,1] \to [0,1]^2$ be given by

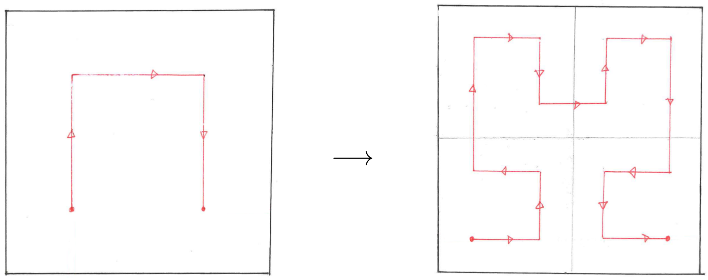
$$f_1(t) = \begin{cases} (1/4, 2t) & \text{for } 0 \leqslant t \leqslant 3/8 \\ (2t - 1/2, 3/4) & \text{for } 3/8 \leqslant t \leqslant 5/8 \\ (3/4, 2 - 2t) & \text{for } 5/8 \leqslant t \leqslant 7/8 \\ (2t - 1, 1/4) & \text{for } 7/8 \leqslant t \leqslant 1 \end{cases} \tag{2}$$

(see Figure 1(a)). Apply the rule in Figure 2 to arrive at the function f_2 depicted in Figure 1(b). Next, divide the unit square into four smaller squares of equal



(a) f_1 (b) f_2 (c) f_3

Figure 1: The first iterations of the Hilbert curve. The function $f_n\colon [0,1] \to [0,1]^2$ is the arc length parametrization of the red curve.



(a) (b)

Figure 2: Each square containing a curve segment of the form (a) is transformed to the form (b) in the next iteration.

size, apply the rule in Figure 2 to each square, and arrive at f_3 depicted in Figure 1(c). Continue in this fashion, resulting in a sequence of continuous functions $f_n\colon [0,1] \to [0,1]^2$ for $n = 1, 2, 3, \ldots$.

Note that the length of the nth curve is 2^n, and that $\|\frac{d}{dt} f_n(t)\|_2 = 2^n$ at all $t \in [0,1]$ (except those t corresponding to kinks in the curve). (Here, $\|\cdot\|_2$ is the Euclidean norm in $\mathbb{R}^2$.)

(a) Try to give a *rough* explanation (less than 1/2 page each) of the following three statements. Feel free to draw pictures instead of giving a written answer.

(i) Let $N \in \mathbb{N}$ and $x, y \in [0,1]^2$. Then

$$\|x - y\|_2 \leqslant 2^{2-N} \tag{3}$$

whenever x and y lie in the same or in neighboring squares with side lengths 2^{-N}.

(ii) Let $n, N \in \mathbb{N}$. Then

$$\|f_n(t) - f_n(s)\|_2 \leqslant 2^{2-N} \qquad \forall\, t, s \in [0,1] \text{ with } |t - s| \leqslant 2^{-2N}. \tag{4}$$

(iii) For every $x \in [0,1]^2$ and $n \in \mathbb{N}$, there is some $t \in [0,1]$ such that

$$\|f_n(t) - x\|_2 \leqslant 2^{-n}. \tag{5}$$

(b) Use (ii) and the Arzela–Ascoli theorem to prove that there is some subsequence $\{f_{n(k)}\}_{k\in\mathbb{N}}$ and some $f \in C([0,1],\mathbb{R}^2)$ such that $f_{n(k)} \to f$ uniformly as $k \to \infty$.

(c) Use (iii) to prove that f is space-filling.

Solution:

(a)

(i) The maximum distance between any pair of points located somewhere in two neighboring squares of side lengths 2^{-N} is $2^{-N}\sqrt{5} < 2^{2-N}$.

(ii) Since $\|\frac{d}{dt}f_n(t)\|_2 = 2^n$, the function f_n is Lipschitz continuous with constant 2^n, so

$$\|f_n(t) - f_n(s)\|_2 \leqslant 2^n|t-s| \leqslant 2^{n-2N}$$

if $|t-s| \leqslant 2^{-2N}$. Hence, if $n \leqslant N$ then

$$\|f_n(t) - f_n(s)\|_2 \leqslant 2^{-N} \leqslant 2^{2-N}.$$

On the other hand, if $n \geqslant N$ and $|t-s| \leqslant 2^{-2N}$ then $f_n(t)$ and $f_n(s)$ lie in the same or neighboring squares of side lengths 2^{-N}, so by (i), we get $\|f_n(t) - f_n(s)\|_2 \leqslant 2^{2-N}$ also in this case.

(iii) In going from Figure 2(a) to Figure 2(b), the distance between the curve and the point in the square which is furthest away from the curve decreases by a factor 1/2, so at step n it has decreased by a factor 2^{1-n}. That distance is $\frac{\sqrt{2}}{4}$ in the first iteration f_1 (the points furthest away being the corners). Hence, for any fixed $x \in [0,1]^2$ we get

$$\min_{t\in[0,1]} \|f_n(t) - x\|_2 \leqslant \frac{\sqrt{2}}{4} \cdot 2^{1-n} = 2^{-1/2-n} < 2^{-n}.$$

(b) Let $\mathcal{F} = \{f_n : n \in \mathbb{N}\} \subset C([0,1],\mathbb{R}^2)$. Then $\mathcal{F}$ is bounded, since all f_n take values in $[0,1]^2$, and it is equicontinuous, by (ii). Hence, by Arzela–Ascoli, there is some subsequence $\{f_{n(k)}\}_k$ converging to some $f \in C([0,1],\mathbb{R}^2)$.

(c) Let $x \in [0,1]^2$. For each n, let $t_n \in [0,1]$ be such that $\|f_n(t_n) - x\|_2 \leqslant 2^{-n}$. Since $t_{n(k)} \in [0,1]$ for all k, and $[0,1]$ is compact, there is some further subsequence of $\{n(k)\}_k$, say, $\{m(k)\}_k$, such that $\{t_{m(k)}\}_k$ converges to some $t \in [0,1]$. Then

$$\begin{aligned}\big\|f(t) - x\big\|_2 &\leqslant \big\|f(t) - f(t_{m(k)})\big\|_2 + \big\|f(t_{m(k)}) - f_{m(k)}(t_{m(k)})\big\|_2 + \big\|f_{m(k)}(t_{m(k)}) - x\big\|_2 \\ &\leqslant \big\|f(t) - f(t_{m(k)})\big\|_2 + \big\|f - f_{m(k)}\big\|_\infty + 2^{-m(k)} \to 0\end{aligned}$$

as $n \to \infty$ (the first term goes to 0 since f is continuous, and the second term goes to zero since $f_{m(k)} \to f$ as $k \to \infty$). Hence, $\|f(t) - x\|_2 = 0$, that is, $f(t) = x$. We conclude that f is surjective, whence it is space-filling.

Problem 3. Let U be a Banach space and let $A \in \mathcal{L}(U)$ (a bounded linear operator from U to U). Our goal is to solve the *vector valued* ordinary differential equation

$$\begin{cases} \frac{d}{dt}u(t) = Au(t) & \text{for } t > 0, \\ u(0) = u_0 \end{cases} \tag{6}$$

for some given $u_0 \in U$, and where $u : [0, \infty) \to U$ is the unknown function. Here, the derivative is defined in the usual sense,

$$\frac{d}{dt}u(t) = \lim_{h \to 0} \tfrac{1}{h}\big(u(t+h) - u(t)\big).$$

Recall that if A is just a number, then the solution of (6) is $u(t) = u_0 e^{At}$. Based on this intuition we will define the exponential function for linear operators, and prove that it provides a solution for (6).

(a) Define the function $\exp\colon \mathcal{L}(U) \to \mathcal{L}(U)$ by

$$\exp(B) = \sum_{k=0}^{\infty} \tfrac{1}{k!}B^k \qquad \text{for } B \in \mathcal{L}(U). \tag{7}$$

Prove that exp is well defined, and that

$$\big\|\exp(B)\big\|_{\mathcal{L}} \leqslant e^{\|B\|_{\mathcal{L}}}. \tag{8}$$

Note:

- We use the convention that $B^0 = I_U$ for all $B \in \mathcal{L}(U)$.
- By *well defined*, we mean that you need to prove that the series (7) always converges, and that $\exp(B) \in \mathcal{L}(U)$ for all $B \in \mathcal{L}(U)$.

(b) Prove that $\exp(0_U) = I_U$. (Here, $0_U\colon U \to U$ is the zero operator, defined by $0_U(u) = 0$ for all $u \in U$.)

(c) Define $u(t) = \exp(tA)u_0$ (that is, the bounded linear operator $\exp(tA)$ applied to u_0). Prove that u solves (6).

Solution:

(a) Since U is complete, it is enough to show that the series is absolutely convergent, by Proposition 5.2.3. We have

$$\sum_{k=0}^{\infty}\Big\|\tfrac{1}{k!}B^k\Big\|_{\mathcal{L}} \leqslant \sum_{k=0}^{\infty}\frac{1}{k!}\|B\|_{\mathcal{L}}^k = e^{\|B\|_{\mathcal{L}}} < \infty$$

for all $B \in \mathcal{L}(U)$, so the series in (7) is indeed absolutely convergent. Hence, it converges to some element of $\mathcal{L}(U)$. Moreover,

$$\left\|\exp(B)\right\|_{\mathcal{L}} \leqslant \sum_{k=0}^{\infty}\left\|\tfrac{1}{k!}B^k\right\|_{\mathcal{L}} \leqslant e^{\|B\|_{\mathcal{L}}}.$$

(b) We get

$$\exp(0_U) = \sum_{k=0}^{\infty} 0_U^k = 0_U^0 + \sum_{k=1}^{\infty} 0_U = I_U.$$

(c) First of all,

$$u(0) = e^{0A}u_0 = e^{0_U}u_0 = I_U u_0 = u_0,$$

so u satisfies the initial data in (6).

For every $t, h \in \mathbb{R}$ with $h \neq 0$ we have

$$\exp((t+h)A) - \exp(tA) = \sum_{k=0}^{\infty}\frac{1}{k!}\big((t+h)A\big)^k - \sum_{k=0}^{\infty}\frac{1}{k!}(tA)^k.$$

For every $N \in \mathbb{N}$ we have

$$\begin{aligned}
&\sum_{k=0}^{N}\frac{1}{k!}\big((t+h)A\big)^k - \sum_{k=0}^{N}\frac{1}{k!}(tA)^k \\
&\quad = \sum_{k=0}^{N}\left(\frac{1}{k!}\big((t+h)A\big)^k - \frac{1}{k!}(tA)^k\right) \\
&\quad = \sum_{k=0}^{N}\frac{1}{k!}\big((t+h)^k - t^k\big)A^k \\
&\quad = \sum_{k=0}^{N}\frac{t^k + kt^{k-1}h + \binom{k}{2}t^{k-2}h^2 + \cdots + h^k - t^k}{k!}A^k \\
&\quad = \sum_{k=0}^{N}\frac{kt^{k-1}h}{k!}A^k + \sum_{k=0}^{N}\frac{\binom{k}{2}t^{k-2}h^2 + \cdots + h^k}{k!}A^k \\
&\quad = hA\sum_{k=1}^{N}\frac{t^{k-1}}{(k-1)!}A^{k-1} + h^2\sum_{k=0}^{N}\frac{\binom{k}{2}t^{k-2} + \cdots + h^{k-2}}{k!}A^k \\
&\quad = hA\sum_{k=0}^{N-1}\frac{t^k}{k!}A^k + h^2\sum_{k=0}^{N}\frac{\binom{k}{2}t^{k-2} + \cdots + h^{k-2}}{k!}A^k.
\end{aligned}$$

Hence, letting $N \to \infty$ and dividing by h gives

$$\begin{aligned}\tfrac{1}{h}\big(\exp((t+h)A) - \exp(tA)\big) &= A\sum_{k=0}^{\infty}\frac{t^k}{k!}A^k + h\sum_{k=0}^{\infty}\frac{\binom{k}{2}t^{k-2} + \cdots + h^{k-2}}{k!}A^k \\ &= A\exp(tA) + h\sum_{k=0}^{\infty}\frac{\binom{k}{2}t^{k-2} + \cdots + h^{k-2}}{k!}A^k.\end{aligned}$$

As $h \to 0$, the second term goes to 0. We conclude that

$$\frac{d}{dt}u(t) = A\exp(tA)u_0 = Au(t).$$

Hints

> **Warning:** Don't use these hints blindly! When writing your solution, *do not assume that the person who will correct your assignment has read the hints.*

Problem 1. If f were differentiable then you could integrate (1) by parts (what is the antiderivative of $\cos(nt)$?). Use Weierstrass' approximation theorem to approximate f by a polynomial.

Problem 2. The expression (2) is for definiteness only. There is no need to manipulate expressions like (2) to solve this problem.

(b) The Arzela–Ascoli theorem can be applied also to sets of functions $\mathcal{F} \subset C(X, \mathbb{R}^2)$ (for a compact set X), just replace the supremum metric by

$$\rho(f, g) = \sup_{x \in X} \|f(x) - g(x)\|_2 \qquad \text{for } f, g \in C(X, \mathbb{R}^2).$$

(The proof of this variant of the theorem consists of simply applying Arzela–Ascoli to both components of the functions.)