METRIC SPACES

Recall:

A sequence $\{x_n\}_n$ in $\mathbb{R}$ <u>converges</u> to $x \in \mathbb{R}$ if for every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon \quad \text{whenever } n \geq N.$$

A function $f : \mathbb{R} \to \mathbb{R}$ is <u>continuous</u> at x if for all $\varepsilon > 0$ there is some $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{when } y \text{ is such that } |x-y| < \delta$$

We want to generalize from $\mathbb{R}$ to other sets

A *metric space* (X, d) consists of a set X and a *metric* d – a function $d : X \times X \to \mathbb{R}$ satisfying:

(i) *Positivity*: $d(x, y) \geq 0 \quad \forall\, x, y \in X$, and

$d(x, y) = 0$ *only* when $x = y$

(ii) *Symmetry*: $d(x, y) = d(y, x) \quad \forall\, x, y \in X$

(iii) *Triangle inequality*: $d(x, y) \leq d(x, z) + d(z, y) \quad \forall\, x, y, z \in X$.

You should think of $d(x, y) =$ "distance between x and y"

Example

$X = \mathbb{R}, \quad d(x,y) = |x-y|$

Example

$X = \mathbb{R}^n, \quad d(x,y) = \|x-y\|$

Example

If V is a vector space and $\|\cdot\|$ is a norm on V, then

$$X = V, \quad d(x,y) = \|x-y\|$$

is a metric space ("the metric induced by $\|\cdot\|$")

Let

$X =$ 

and

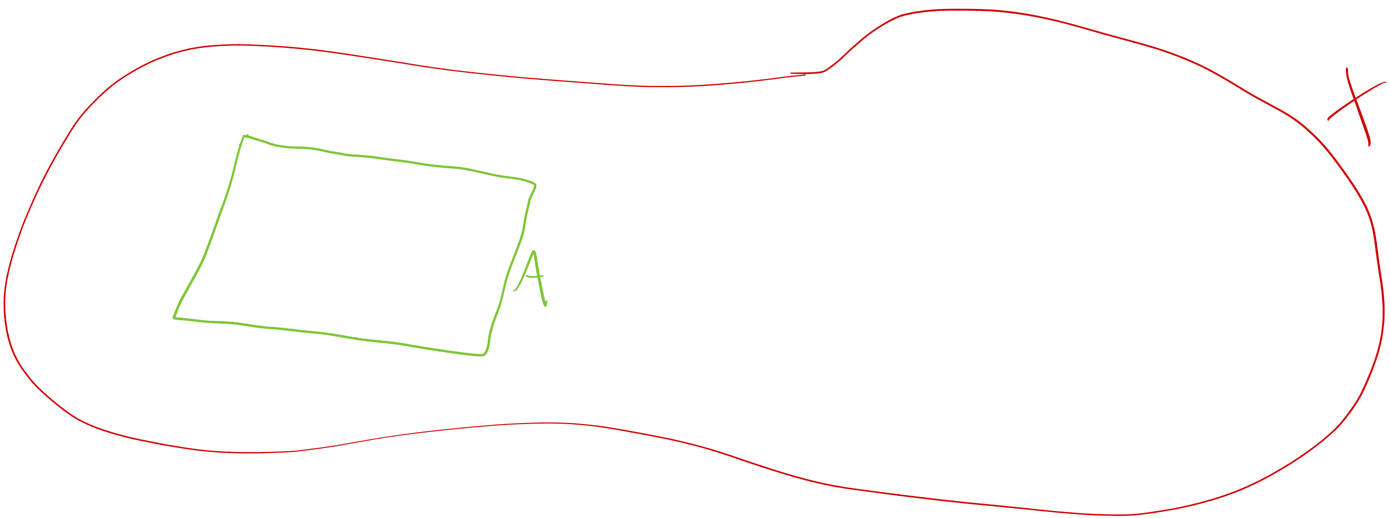
$d(x,y) =$ distance between x and y in the real world

Note: This metric is not induced by any norm

Exercise: Argue *in words* that this is a metric space.

If (X, d) is a metric space and $A \subseteq X$, then (A, d_A) is also a metric space, where

$$d_A(x,y) = d(x,y) \qquad \forall \ x,y \in A.$$



The inverse triangle inequality

If (X,d) is a metric space then

$$|d(x,y) - d(x,z)| \leq d(y,z) \qquad \forall \; x,y,z \in X$$

Exercise: prove this!

QUESTIONS?

COMMENTS?