CONTINUITY

Recall:

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at x if for all $\varepsilon > 0$ there is some $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \qquad \text{when } y \text{ is such that } |x-y| < \delta$$

Continuity

Let (X, d_X) and (Y, d_Y) be metric spaces.
A function $f: X \to Y$ is <u>continuous at $x \in X$</u> if for all $\varepsilon > 0$ there is some $\delta > 0$ such that

$$d_Y(f(x), f(y)) < \varepsilon \quad \text{when } y \in X \text{ is such that } d_X(x, y) < \delta.$$

f is <u>continuous</u> if it is continuous at all $x \in X$.

<u>Recall</u>:

In a metric space (X, d_X):

$$B_X(a; r) = \{ z \in X \ : \ d_X(a, z) < r \}$$

Continuity (equivalent definition)

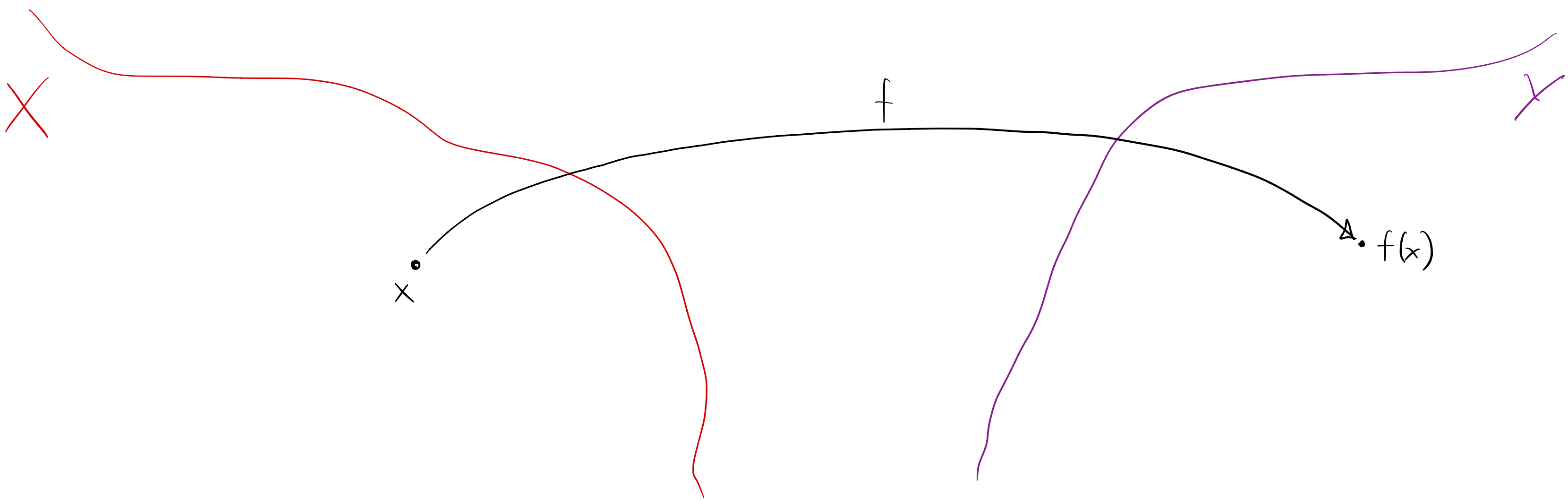
A function $f: X \to Y$ is continuous at $x \in X$ if for every $\varepsilon > 0$ there is some $\delta > 0$ such that

$$f(y) \in B_Y(f(x);\varepsilon) \quad \text{for every } y \in B_X(x;\delta)$$

Continuity (equivalent definition)

A function $f: X \to Y$ is continuous at $x \in X$ if for every $\varepsilon > 0$ there is some $\delta > 0$ such that

$$f\big(B_X(x;\delta)\big) \subseteq B_Y\big(f(x);\varepsilon\big)$$



Continuity (equivalent definition)

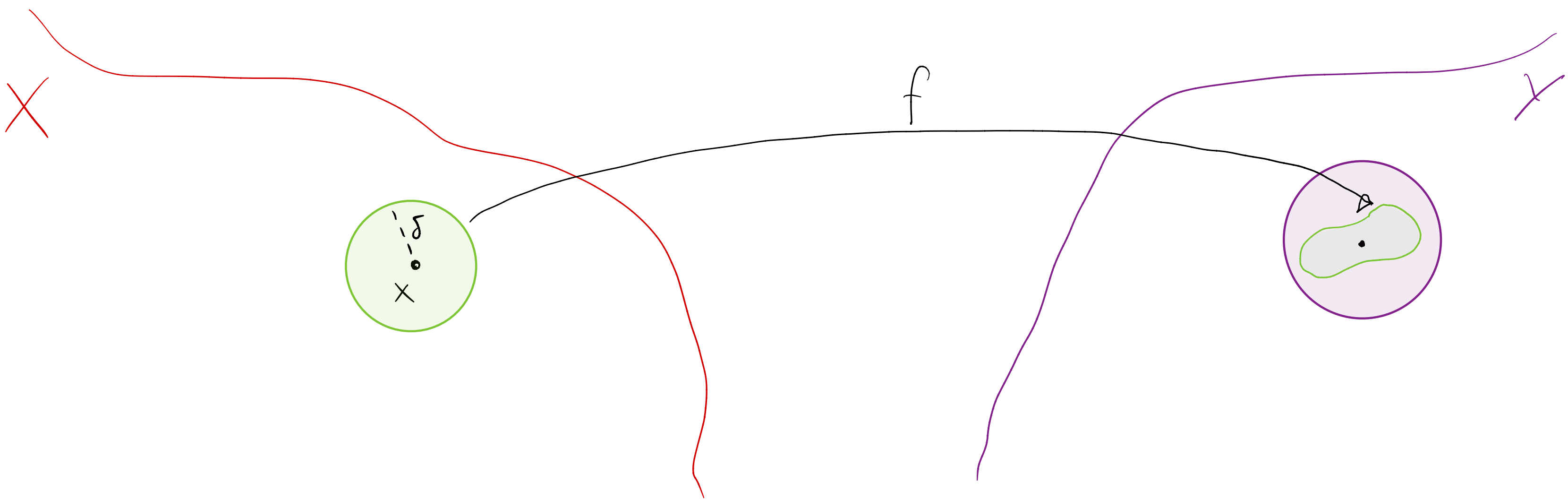
A function $f : X \to Y$ is <u>continuous at $x \in X$</u> if for every $\varepsilon > 0$ there is some $\delta > 0$ such that

$$f\big(B_X(x;\delta)\big) \subseteq B_Y\big(f(x);\varepsilon\big)$$

Continuity (equivalent definition)

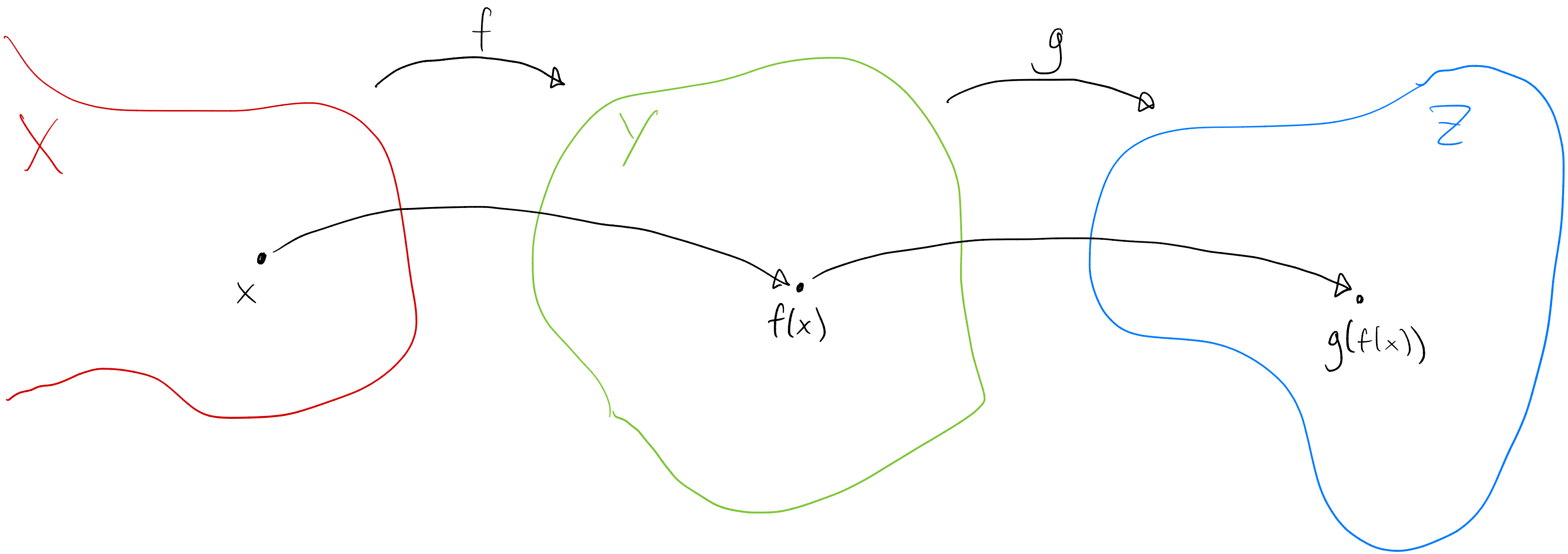
A function $f: X \to Y$ is <u>continuous at $x \in X$</u> if for every $\varepsilon > 0$ there is some $\delta > 0$ such that

$$f\big(B_X(x;\delta)\big) \subseteq B_Y\big(f(x);\varepsilon\big)$$



Proposition:

If $f: X \to Y$ and $g: Y \to Z$ are continuous, then so is $g \circ f$.



Sequential continuity

Let $f : X \to Y$ and $x \in X$ be given. Then TFAE:

(i) f is continuous at x

(ii) if $\{x_n\}_n$ is any sequence in X converging to x, then also

$$f(x_n) \longrightarrow f(x) \quad \text{as } n \to \infty.$$

Sequential continuity

Let $f: X \to Y$ and $x \in X$ be given. Then TFAE:

(i) f is continuous at x

(ii) if $\{x_n\}_n$ is any sequence in X converging to x, then also

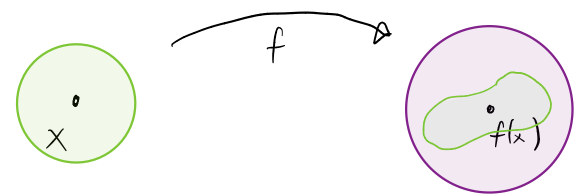
$$f(x_n) \longrightarrow f(x) \quad \text{as} \quad n \to \infty.$$

<u>Proof of (i) $\Rightarrow$ (ii):</u> Let $\varepsilon > 0$.

Let $\delta > 0$ be such that $d_Y(f(x), f(y)) < \varepsilon$ when $d_X(x, y) < \delta$.

Let $N \in \mathbb{N}$ be s.t. $d_X(x, x_n) < \delta$ when $n \geq N$.

Then also $d_Y(f(x), f(x_n)) < \varepsilon$.



Sequential continuity

Let $f : X \to Y$ and $x \in X$ be given. Then TFAE:

(i) f is continuous at x

(ii) if $\{x_n\}_n$ is any sequence in X converging to x, then also

$$f(x_n) \longrightarrow f(x) \quad \text{as} \quad n \to \infty.$$

Proof of (ii) $\Rightarrow$ (i): Assume f is discontinuous. Then there is some $\varepsilon > 0$ such that for any $\delta > 0$ there is some $y \in B_X(x;\delta)$ such that $d_Y(f(x), f(y)) \geq \varepsilon$. For every $\delta = \frac{1}{n}$ $(n \in \mathbb{N})$, let $x_n = y$ be such a point. Then $d_X(x_n, x) < \frac{1}{n} \xrightarrow[n\to\infty]{} 0$, but

$$d_Y\big(f(x), f(x_n)\big) \geq \varepsilon \;\not\xrightarrow[n\to\infty]{}\; 0.$$

$\blacksquare$

QUESTIONS?
COMMENTS?