ALTERNATIVE DEFINITIONS OF CONTINUITY

<u>Continuity</u> is no important that it's useful to have several (equivalent) definitions of it.

Definition 1

Let (X, d_X) and (Y, d_Y) be metric spaces.
A function $f: X \longrightarrow Y$ is continuous at $x \in X$ if for all $\varepsilon > 0$ there is some $\delta > 0$ such that

$$d_Y(f(x), f(y)) < \varepsilon \qquad \text{when } y \in X \text{ is such that } d_X(x, y) < \delta.$$
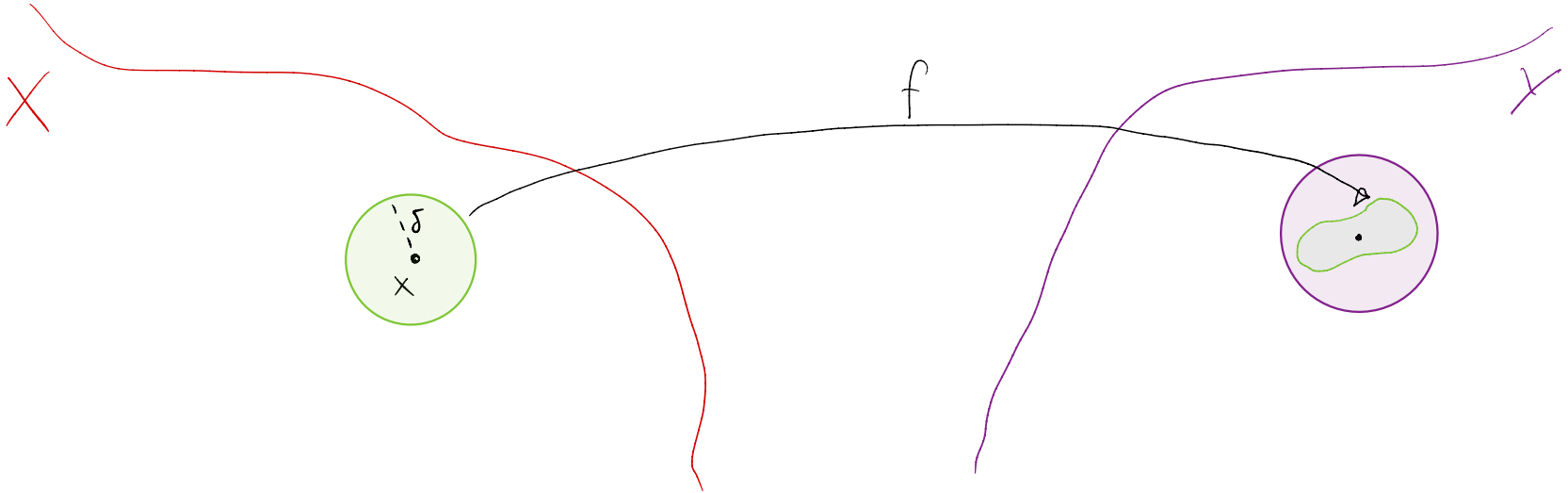
Definition 2

A function $f: X \longrightarrow Y$ is <u>continuous at $x \in X$</u> if for every $\varepsilon > 0$ there is some $\delta > 0$ such that

$$f(y) \in B_Y(f(x); \varepsilon) \qquad \text{for every } y \in B_X(x; \delta)$$
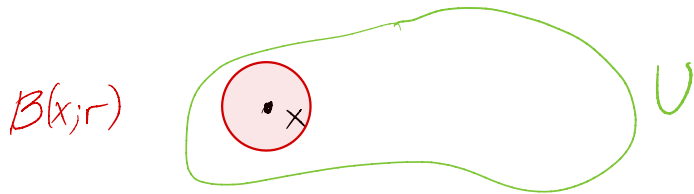
Definition 3

A function $f : X \longrightarrow Y$ is <u>continuous at $x \in X$</u> if for every $\varepsilon > 0$ there is some $\delta > 0$ such that

$$f\Big(B_X(x; \delta)\Big) \subseteq B_Y\Big(f(x); \varepsilon\Big)$$

X

f

Y

δ

x

A __neighborhood__ of a point $x \in X$ is a set $U \subseteq X$ for which there is some $r > 0$ such that $B(x; r) \subseteq U$. (NO: omega)

$B(x; r)$

U

__Note__:
- $B(x; r)$ is a neighborhood of x
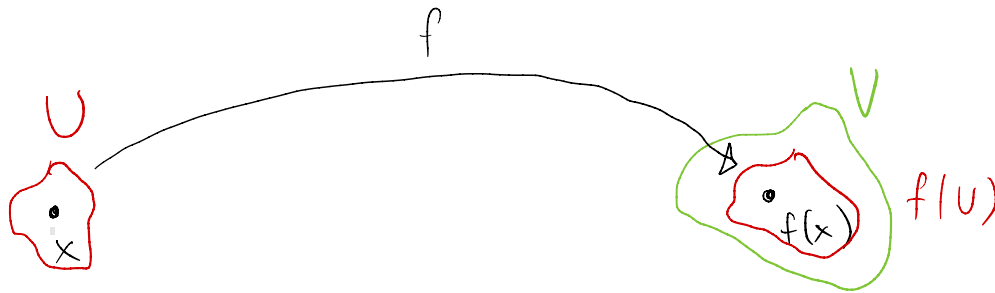- A set U is open __iff__ it is a neighborhood of all of its members.

Definition 4

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \to Y$.

Then TFAE:

(i) f is continuous

(ii) For every $x \in X$ and nbd. V of $f(x)$, there is a nbd. U of x so that $f(U) \subseteq V$.

Definition 4

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \to Y$. Then TFAE:

(i) f is continuous

(ii) For every $x \in X$ and nbd. V of $f(x)$, there is a nbd. U of x so that $f(U) \subseteq V$.

Recall: TFAE:

(i) f is continuous
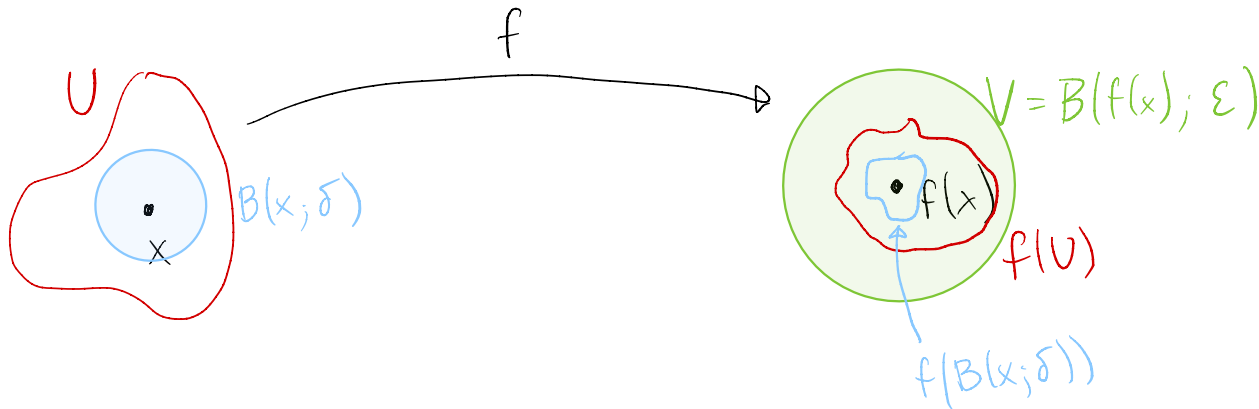
(iî) for every $x \in X$ and $\varepsilon > 0$ there is some $\delta > 0$ such that $f(B(x; \delta)) \subseteq B(f(x); \varepsilon)$

$\underline{(ii) \Rightarrow (\hat{ii})}$: Let $\varepsilon > 0$. Then $V = B(f(x), \varepsilon)$ is a nbd. of $f(x)$,

so there is some nbd. U of x so that

$$f(U) \subseteq B(f(x); \varepsilon).$$

Let $\delta > 0$ be such that $B(x; \delta) \subseteq U$. Then

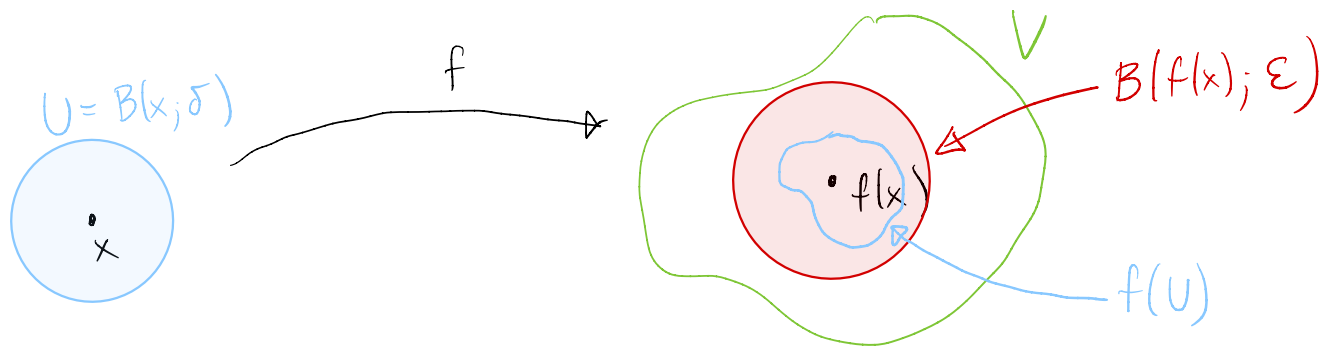$$f(B(x; \delta)) \subseteq f(U) \subseteq B(f(x); \varepsilon).$$

$\underline{(ii) \Longleftarrow (\hat{ii})}$: Let $x \in X$ and let V be a nbd. of $f(x)$.

Let $\varepsilon > 0$ be such that
$$B(f(x); \varepsilon) \subseteq V.$$

There is some $\delta > 0$ such that
$$f(B(x; \delta)) \subseteq B(f(x); \varepsilon).$$

If $U = B(x; \delta)$ then also $f(U) \subseteq B(f(x); \varepsilon) \subseteq V.$

Definition 5

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \to Y$. Then TFAE:

(i) f is continuous

(ii) if $V \subseteq Y$ is open then $f^{-1}(V)$ is open.

(iii) if $F \subseteq Y$ is closed then $f^{-1}(F)$ is closed.

Continuous functions are those whose inverse image preserves openness / closedness.

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \to Y$.

Then TFAE:

(i) f is continuous

(ii) if $V \subseteq Y$ is open then $f^{-1}(V)$ is open.

(iii) if $F \subseteq Y$ is closed then $f^{-1}(F)$ is closed.

(i) $\Rightarrow$ (ii): Let $V \subseteq Y$ be open. If $f^{-1}(V) = \emptyset$ then we are done. Otherwise, let $x \in f^{-1}(V)$. Then V is a nbd. of $f(x)$, so there is a nbd. U of x such that $f(U) \subseteq V$. Then also $U \subseteq f^{-1}(V)$. Last, let $r > 0$ be such that $B(x; r) \subseteq U$. Then also $B(x; r) \subseteq f^{-1}(V)$, so $f^{-1}(V)$ is open.

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \to Y$.

Then TFAE:

(i) f is continuous
(ii) if $V \subseteq Y$ is open then $f^{-1}(V)$ is open.
(iii) if $F \subseteq Y$ is closed then $f^{-1}(F)$ is closed.

(i) $\Leftarrow$ (ii): Let $x \in X$, let $\varepsilon > 0$, let $V = B(f(x); \varepsilon)$. Then $f^{-1}(V)$ is open, and $x \in f^{-1}(V)$, so there is some $\delta > 0$ so that $B(x; \delta) \subseteq f^{-1}(V)$. Then also

$$f(B(x; \delta)) \subseteq f(f^{-1}(V)) \subseteq V = B(f(x); \varepsilon).$$

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \to Y$.

Then TFAE:

(i) f is continuous

(ii) if $V \subseteq Y$ is open then $f^{-1}(V)$ is open.

(iii) if $F \subseteq Y$ is closed then $f^{-1}(F)$ is closed.

(ii) $\Rightarrow$ (iii): Let $F \subseteq Y$. Then:

F is closed $\Leftrightarrow$ F^c is open $\Rightarrow$ $f^{-1}(F^c)$ is open

$\Leftrightarrow$ $f^{-1}(F)^c$ is open (as $f^{-1}(F^c) = f^{-1}(F)^c$)

$\Leftrightarrow$ $f^{-1}(F)$ is closed.

Example:
Let $f: \mathbb{R} \to \mathbb{R}$ be continuous. Show that

$$A = \{x : f(x) > 0\}$$

is open.

We can write $A = f^{-1}((0, \infty))$. The set $(0, \infty)$ is open, so A must be open.

QUESTIONS?
COMMENTS?