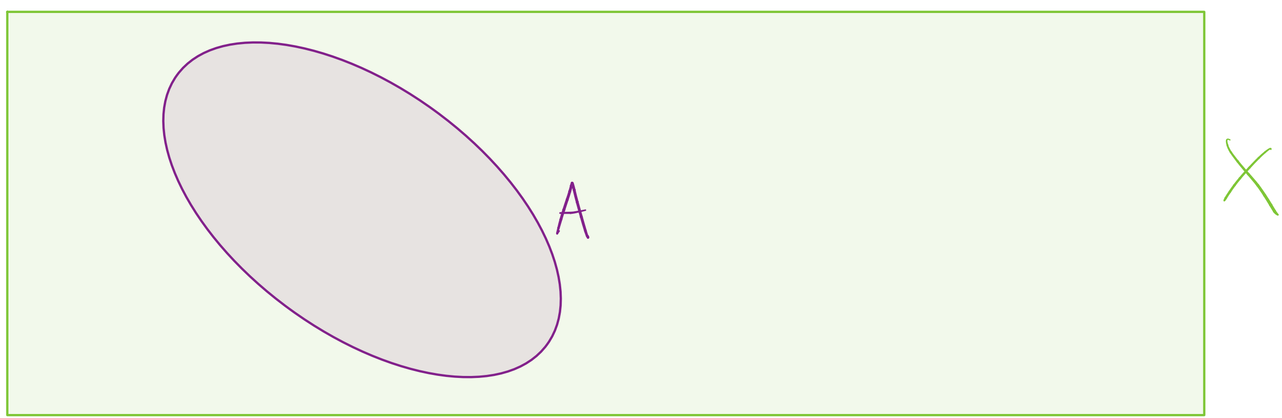
OPEN AND CLOSED SETS

Interior/exterior/boundary points

Let X be a set and $A \subseteq X$.

Intuitively, a point x is either inside of A, outside of A, or on the boundary of A.



Recall:

The open ball centred at $x \in X$ with radius $r \geqslant 0$ is

$$B(x;r) = \{ z \in X : d(x,z) < r \}$$

The closed ball centred at $x \in X$ with radius $r \geq 0$ is

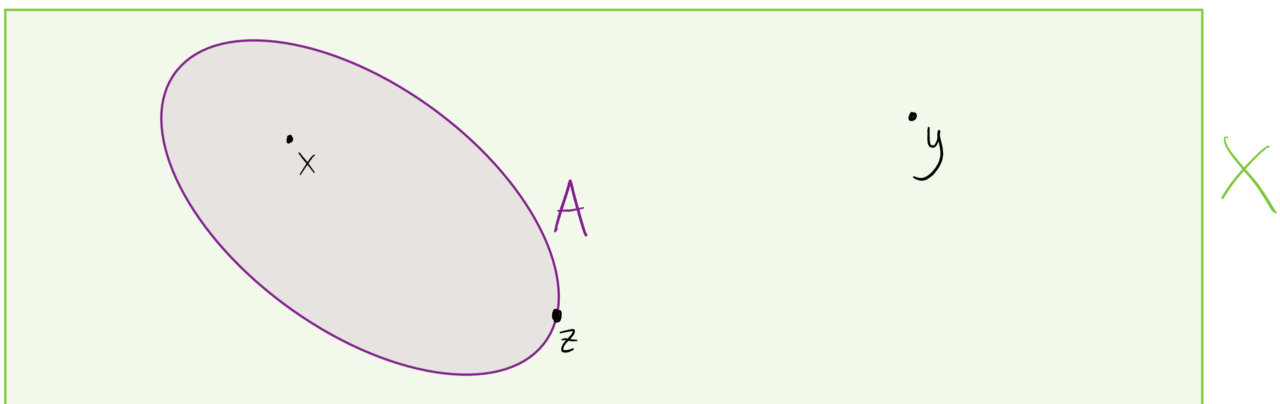
$$B(x;r) = \{ z \in X : d(x,z) \leq r \}$$

Let (X, d) be a metric space and let $A \subseteq X$.

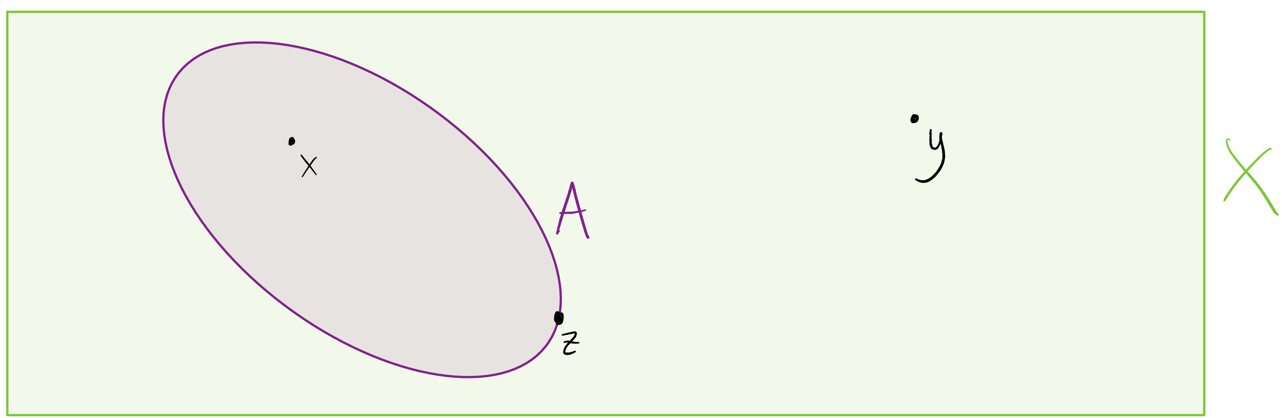
- $x \in X$ is an <u>interior point of A</u> if $B(x; r) \subseteq A$ for some $r > 0$
- $y \in X$ is an <u>exterior point of A</u> if $B(y; r) \subseteq A^c$ for some $r > 0$
- $z \in X$ is a <u>boundary point of A</u> if it is neither of the above:

  both $B(z; r) \cap A \neq \emptyset$ and $B(z; r) \cap A^c \neq \emptyset$ for every $r > 0$

<u>Note</u>: Every point in X is exactly <u>one</u> of these three.



- $A^\circ = \operatorname{int} A = \{\text{all interior points of } A\}$
  is the <u>interior of A</u> (NO: <u>det indre</u> av A)
- $\partial A = \{\text{all boundary points of } A\}$
  is the <u>boundary of A</u> (NO: <u>randa til A</u>)
- $\overline{A} = \operatorname{cl} A = A \cup \partial A = ((A^c)^\circ)^c$
  is the <u>closure of A</u> (NO: <u>tillukningen til A</u>)



<u>Exercise:</u>

For any $A \subseteq X$, we have $\partial A = \partial(A^c)$

Let (X, d) be a metric space.

$A \subseteq X$ is <u>open</u> if $A = A^\circ$

$A \subseteq X$ is <u>open</u> if A contains none of its boundary points.

$A \subseteq X$ is <u>open</u> if $A \cap \partial A = \emptyset$.

$A \subseteq X$ is <u>open</u> if $\forall\ x \in A$ there is some $r > 0$ s.t. $B(x; r) \subseteq A$.

Let (X, d) be a metric space.

$B \subseteq X$ is <u>closed</u> if $B = \overline{B}$

$B \subseteq X$ is <u>closed</u> if B contains all of its boundary points

$B \subseteq X$ is <u>closed</u> if $\partial B \subseteq B$.

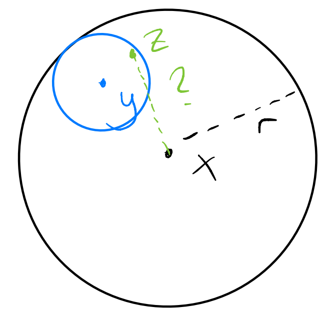
Exercise:

Some set $A \subseteq X$ is open iff A^c is closed.

Some set $B \subseteq X$ is closed iff B^c is open.

Hint: Use the previous exercise.

The open ball $B(x;r) = \{y \in X : d(x,y) < r\}$ is open.
The closed ball $\overline{B}(x;r) = \{y \in X : d(x,y) \leq r\}$ is closed.



Proof: Recall that a set B is open iff $\forall\, x \in B$ $\exists\, r > 0$ such that $B(x;r) \subseteq B$.

Let $y \in B(x;r)$.

Let $s = r - d(x,y)$. Then $s > 0$, and $B(y;s) \subseteq B(x;r)$.

Indeed, $z \in B(y;s) \Leftrightarrow d(y,z) < s$, so

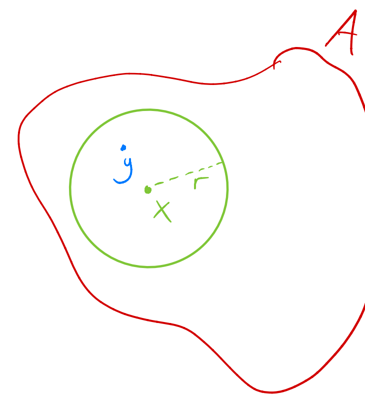
$$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + r - d(x,y) = r.$$

$\blacksquare$

Let $A \subseteq X$. Then A° is open, and $\bar{A}$ is closed.

<u>Proof</u>: Recall that $A^\circ = \{\text{all interior points of } A\}$.

Let $x \in A^\circ$. Then x is an interior point of A, so $\exists\, r > 0$ such that $B(x;r) \subseteq A$.



<u>Claim</u>: $B(x;r) \subseteq A^\circ$.

Let $y \in B(x;r)$. Then $d(x,y) < r$, so

$$B(y;s) \subseteq B(x;r) \subseteq A$$

if $s = r - d(x,y)$. Hence, $y \in A^\circ$. ■

<u>Exercise</u>: $\bar{A}$ is closed.

Let $F \subseteq X$. Then TFAE:

(i) F is closed

(ii) if $\{x_n\}_n$ is a sequence in F converging to $x \in X$, then $x \in F$.

<u>(i) $\Rightarrow$ (ii)</u>: If $x \notin F$ then $x \in F^c$, which is open, so $\exists\, r > 0$ such that $B(x;r) \subseteq F^c$.

Let $N \in \mathbb{N}$ such that $x_n \in B(x;r)$ $\forall\, n \geq N$.

Then both $x_n \in F$ and $x_n \in B(x;r) \subseteq F^c$

↯

Let $F \subseteq X$. Then TFAE:

(i) F is closed

(ii) if $\{x_n\}_n$ is a sequence in F converging to $x \in X$, then $x \in F$.

(i) $\Leftarrow$ (ii): Let $x \in \partial F$. Then both $B(x;r) \cap F \neq \emptyset$ and $B(x;r) \cap F^c \neq \emptyset$ for every $r > 0$.

Then for every $n \in \mathbb{N}$, there is some $x_n \in B(x; \frac{1}{n}) \cap F$.

Moreover, $d(x_n, x) < \frac{1}{n} \xrightarrow[n \to \infty]{} 0$.

Hence, $x_n \xrightarrow[n \to \infty]{} x$, so by (ii), $x \in F$.

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QUESTIONS?

COMMENTS?