COMPLETENESS

Recall: A sequence $\{x_n\}_n$ in $\mathbb{R}$ is Cauchy if for every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon \qquad \text{whenever } n, m \geq N.$$

Example:

$x_1 = 1$

$x_2 = 1.4$

$x_3 = 1.41$

$x_4 = 1.414$

$x_5 = 1.4142$

$x_6 = 1.41421$

$x_7 = 1.414213$

$\vdots$

This sequence is Cauchy: If $n, m \geq N$ then $|x_n - x_m| < 10^{-N+1} \xrightarrow[N \to \infty]{} 0.$

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Recall: Every Cauchy sequence in $\mathbb{R}$ converges.

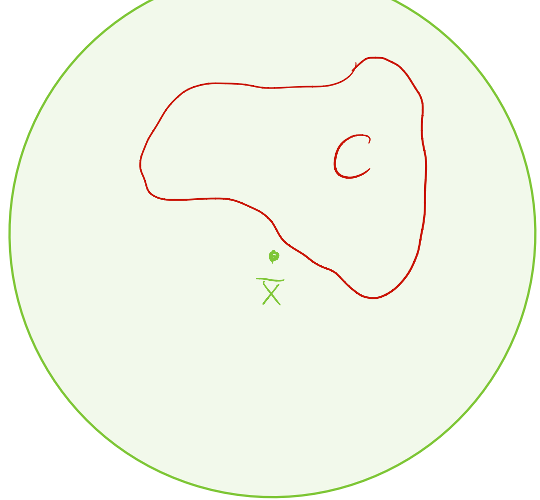
Let (X,d) be a metric space.

A sequence $\{x_n\}_n$ in X is <u>Cauchy</u> if for every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon \quad \text{whenever } n, m \geq N.$$

A subset C of a metric space (X, d) is <u>bounded</u> if there is some $\bar{x} \in X$ and $R > 0$ such that $C \subseteq B(\bar{x}; \cancel{M}\, R)$

A sequence $\{x_n\}_n$ in a metric space (X, d) is <u>bounded</u> if the set $\{x_1, x_2, \dots\}$ is bounded.



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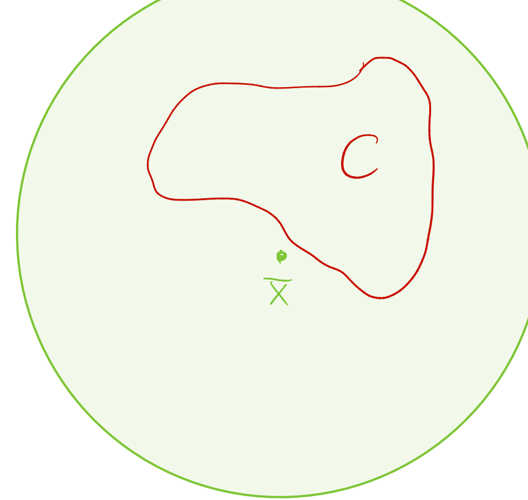

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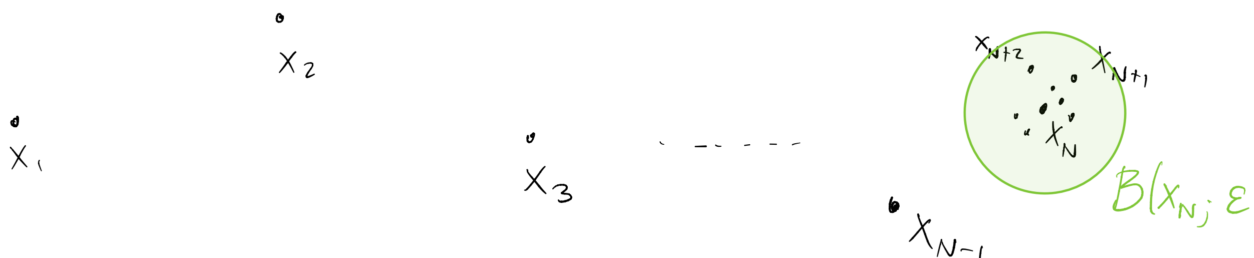
Every Cauchy sequence is bounded.



$\boxed{\text{Every convergent sequence is Cauchy.}}$

Proof:

Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that

$$d(x_n, x) < \varepsilon \qquad \forall \quad n \geq N.$$

If $n, m \geq N$ then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \varepsilon + \varepsilon = 2\varepsilon.$$

$\blacksquare$

A metric space (X,d) is <u>complete</u> if every Cauchy sequence converges.

<u>Examples</u>:

- $X=\mathbb{R}$, $d(x,y)=|x-y|$ is complete
- $X=\mathbb{R}^n$, $d(x,y)=\|x-y\|$ is complete
- If (X,d) is complete then (X^n,d_n) is complete (where $X^n=X\times X\times\cdots\times X$, $d_n(x,y)=d(x_1,y_1)+\cdots+d(x_n,y_n)$)
- $X=C([a,b],\mathbb{R})$, $d_\infty(f,g)=\sup_{t\in[a,b]}|f(t)-g(t)|$ is complete
- $X=C([a,b],\mathbb{R})$, $d_1(f,g)=\int_a^b|f(t)-g(t)|\,dt$ is <u>not</u> complete.

If (X,d) is complete and $A \subseteq X$, then (A,d) is complete ~~iff~~ A is closed.

Proof of "$\Longrightarrow$":

Let $\{x_n\}_n$ be a sequence in A converging to $x \in X$. We want $x \in A$.

Since $\{x_n\}_n$ converges, it is Cauchy, both as a sequence in (X,d) and (A,d). By completeness of (A,d), there is some $y \in A$ such that $x_n \xrightarrow[n\to\infty]{} y$. But also $x_n \xrightarrow[n\to\infty]{} x$, so $x = y \in A$.

If (X,d) is complete and $A \subseteq X$, then (A,d) is complete iff A is closed.

Proof of "$\Longleftarrow$":

Let $\{x_n\}_n$ be a Cauchy sequence in (A,d). Then it is Cauchy in (X,d), so $x_n \xrightarrow[n\to\infty]{} x \in X$. But A is closed, so also $x \in A$.

$\blacksquare$

QUESTIONS?

COMMENTS?

Example: Let $X = C([0,1], \mathbb{R})$ and $d_\infty(f,g) = \sup_{t \in [0,1]} |f(t) - g(t)|$.
Let $f_n(t) = 1 + t + \cdots + \frac{t^n}{n!}$. Is $\{f_n\}_n$ Cauchy in (X, d_∞)?

First, $f_n \in X$, and if $n, m \geq N$ and, say, $n < m$ then

$$d_\infty(f_n, f_m) = \sup_{t \in [0,1]} \left| \sum_{l=0}^{n} \frac{t^l}{l!} - \sum_{l=0}^{m} \frac{t^l}{l!} \right| = \sup_{t \in [0,1]} \left| \sum_{l=n+1}^{m} \frac{t^l}{l!} \right|$$

$$\leq \sup_{t \in [0,1]} \sum_{l=n+1}^{m} \frac{|t|^l}{l!} = \sum_{l=n+1}^{m} \frac{1}{l!} \leq \sum_{l=n+1}^{\infty} \frac{1}{l!}$$

$$\leq \sum_{l=N}^{\infty} \frac{1}{l!} \longrightarrow 0$$

as $N \to \infty$. Hence, $\{f_n\}$ is Cauchy in (X, d_∞). $\square$