COMPACTNESS

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Recall:

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But it's not always so easy to show that a sequence is Cauchy!

- It's too much to require that *all* sequences converge.
- We can require that a *subsequence* will converge.

Let $\{x_n\}_{n\in\mathbb{N}}$ be some sequence. A <u>subsequence</u> is a sequence of the form $\{x_{n(k)}\}_{k\in\mathbb{N}}$, where $n(k)\in\mathbb{N}$ $\forall$ $k\in\mathbb{N}$, and $n(1)<n(2)<\cdots$

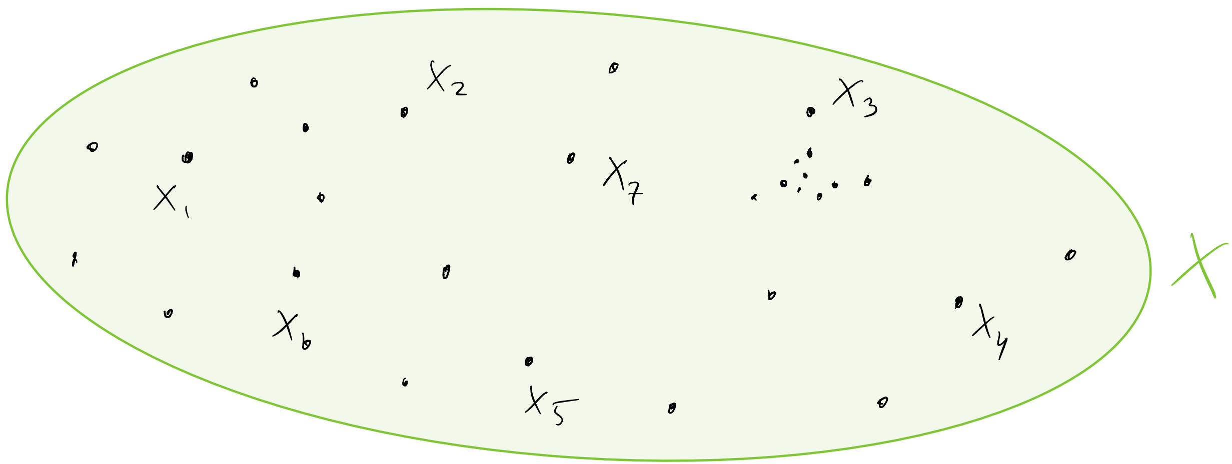
<u>Note</u>: $n(k)\geq k$ $\forall$ $k\in\mathbb{N}$.

Let (X,d) be a metric space. If $\{x_n\}_n$ converges, then all subsequences converge, and the limit is the same.

<u>Proof</u>: Let $\varepsilon>0$, let $N\in\mathbb{N}$ be such that $d(x_n,x)<\varepsilon$ when $n\geq N$. Then $n(k)\geq N$ when $k\geq N$, so also $d(x_{n(k)},x)<\varepsilon$. ∎

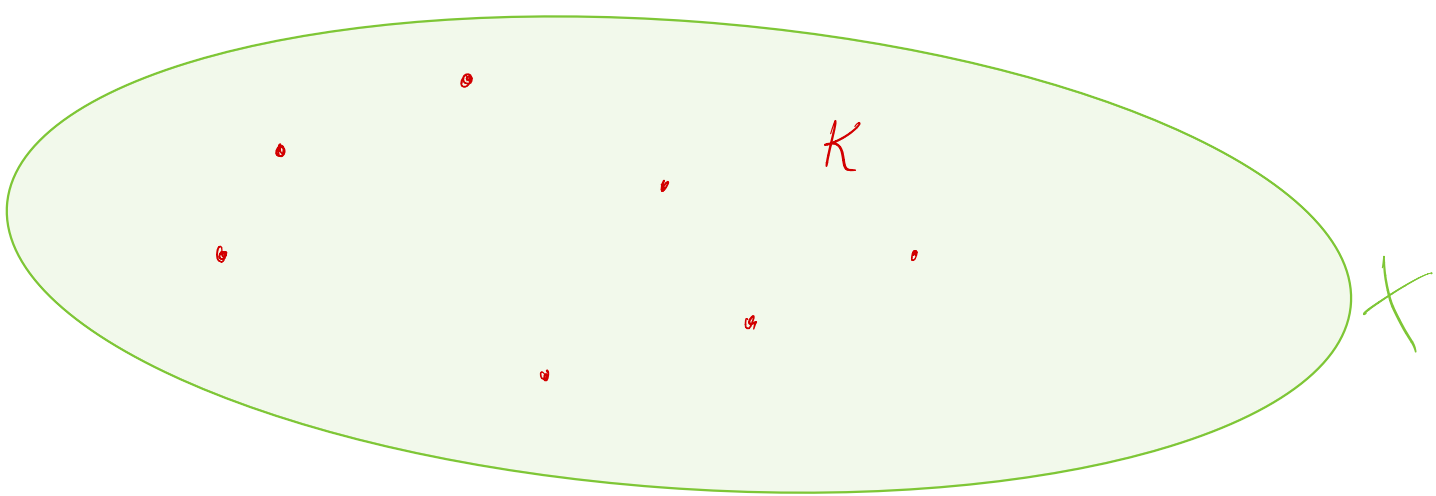
A metric space (X, d) is <u>compact</u> if every sequence $\{x_n\}_n$ has a convergent subsequence $\{x_{n(k)}\}_k$.

A subset K of a metric space (X, d) is <u>compact</u> if (K, d) is compact.



Let (X,d) be a metric space.
If $K \subseteq X$ is finite then it is compact.

Proof: Let $\{x_n\}_n$ be a sequence in K.



Let (X,d) be compact. Then it is complete.

<u>Proof</u>: Let $\{x_n\}_n$ be Cauchy. As (X,d) is compact, there is some convergent subsequence $\{x_{n(k)}\}_k$ converging to $x \in X$.

For $\varepsilon > 0$, let $\tilde{N} \in \mathbb{N}$ be s.t. $d(x_{n(k)}, x) < \varepsilon$ when $k \geq \tilde{N}$, and let $\hat{N}$ be s.t. $d(x_n, x_m) < \varepsilon$ when $n, m \geq \hat{N}$.

Let $M = \max(\hat{N}, n(\tilde{N}))$. If $m \geq M$ then

$$d(x_m, x) \leq \underbrace{d(x_m, x_{n(M)})}_{<\varepsilon} + \underbrace{d(x_{n(M)}, x)}_{<\varepsilon} < 2\varepsilon.$$

Hence, $\{x_n\}_n$ converges to x. ∎

Summary

- Convergence $\implies$ subsequences converge
- Finite $\implies$ compact
- Compact $\implies$ complete

QUESTIONS?

COMMENTS?