COMPACTNESS

II

A metric space (X, d) is <u>compact</u> if every sequence $\{x_n\}_n$ has a convergent subsequence $\{x_{n(k)}\}_k$.

<u>Recall</u>: Finite sets are compact.

But the opposite is not true!

We would like to <u>characterize</u> the compact sets.

If $K \subseteq X$ is compact then it is <u>closed</u> and <u>bounded</u>

<u>Proof</u>: • K is closed: Let $\{x_n\}_n$ be a sequence in K converging to $x \in X$.

K is compact, so there is a subsequence $\{x_{n(k)}\}_k$ converging to some $y \in K$.

But then $\{x_{n(k)}\}_k$ converges to x <u>and</u> y, so $x = y \in K$.

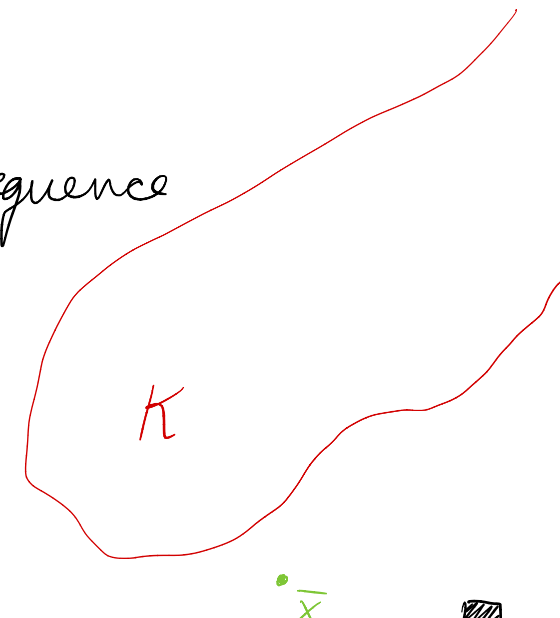
If $K \subseteq X$ is compact then it is closed and bounded

Proof: • K is bounded: Assume not. Fix $\bar{x} \in X$. Then for every $n \in \mathbb{N}$, there is some $x_n \in K$ with
$d(x_n, \bar{x}) \geq n$.

Let $\{x_{n(k)}\}_k$ be a convergent subsequence of $\{x_n\}_n$. Then

$$n(k) \leq d(x_{n(k)}, \bar{x}) \xrightarrow[k\to\infty]{} d(x, \bar{x})$$

$\lightning$



$\blacksquare$

- Thus, compact $\Longrightarrow$ closed, bounded.
- The opposite might not be true!

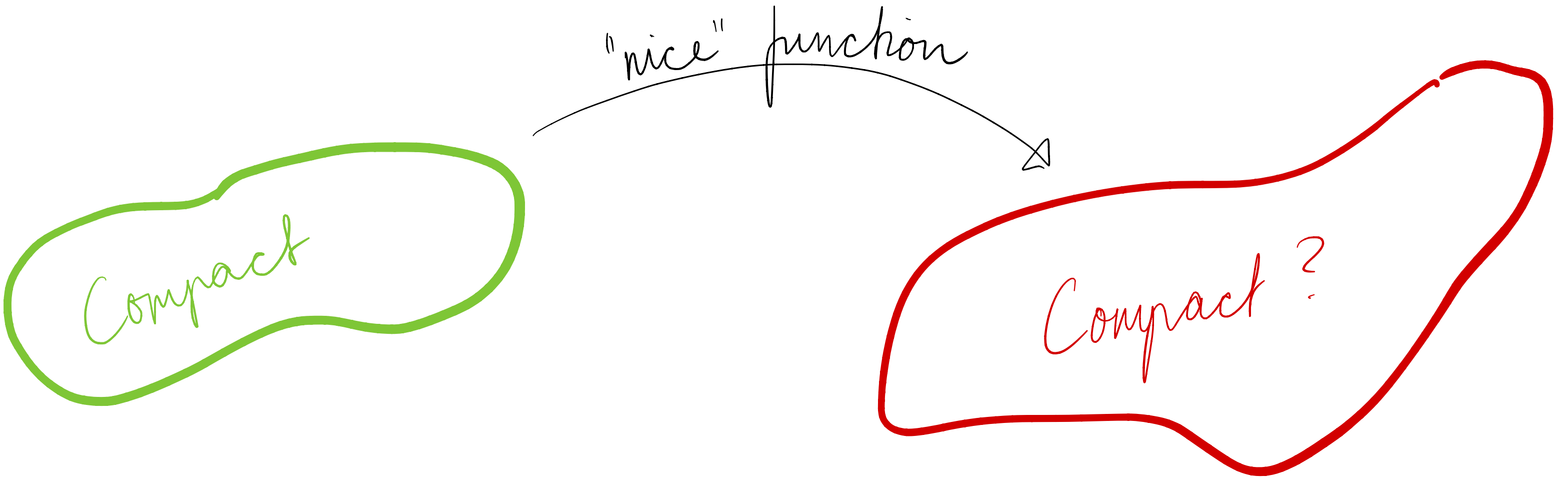
Let $X = \mathbb{R}^n$, $d(x,y) = \|x-y\|$. Let $K \subseteq \mathbb{R}^n$. Then:

K is compact $\Longleftrightarrow$ K is closed and bounded

Proof: We have already shown "$\Longrightarrow$". Let K be closed, bounded. Let $\{x_n\}_n$ be a sequence in K. Then $\{x_n\}_n$ is bounded, so by Bolzano-Weierstrass, there is a subsequence $\{x_{n(k)}\}_k$ and some $x \in \mathbb{R}^n$ s.t. $x_{n(k)} \xrightarrow[k\to\infty]{} x$.

But $\{x_{n(k)}\}_k$ is a sequence in K, which is closed, so $x \in K$.

Hence, K is compact. $\blacksquare$



"nice" function
Compact
Compact ?

Let (X, d_X) and (Y, d_Y) be metric spaces and $f: X \to Y$ continuous. If $K \subseteq X$ is compact then $f(K)$ is compact.

<u>Proof</u>: Let $\{y_n\}_n$ be a sequence in $f(K)$.

Let $x_n \in K$ be s.t. $f(x_n) = y_n$, $\forall\, n \in \mathbb{N}$.

Let $\{x_{n(k)}\}_k$ be a subsequence s.t. $x_{n(k)} \xrightarrow[k \to \infty]{} x \in K$.

f is continuous, so $y_{n(k)} = f(x_{n(k)}) \xrightarrow[k\to\infty]{} f(x) \in f(K)$

$\blacksquare$

<u>Note</u>: • Continuity is essential! If $f(x) = \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$ $(x \in \mathbb{R})$ then $f([-1,1])$ is unbounded, hence noncompact.

• Not true for inverse images! If $f(x) = \sin x$ $(x \in \mathbb{R})$ then $f^{-1}([-1,1]) = \mathbb{R}$, which is noncompact.

Extremal value theorem

Let (X,d) be compact and $f : X \to \mathbb{R}$ continuous.
Then f attains a maximum and a minimum.

<u>Proof</u>: $f(X) \subseteq \mathbb{R}$ is compact, so it is closed and bounded. Hence, both $\inf(f(X))$ and $\sup(f(X))$ lie in $f(X)$.
Thus, there are $a, b \in X$ so that

$$f(a) = \inf(f(X)) \quad \text{and} \quad f(b) = \sup(f(X)).$$

Hence,

$$f(a) \leq f(x) \leq f(b) \qquad \forall \; x \in X.$$

$\blacksquare$

QUESTIONS?

COMMENTS?