COMPACTNESS

III

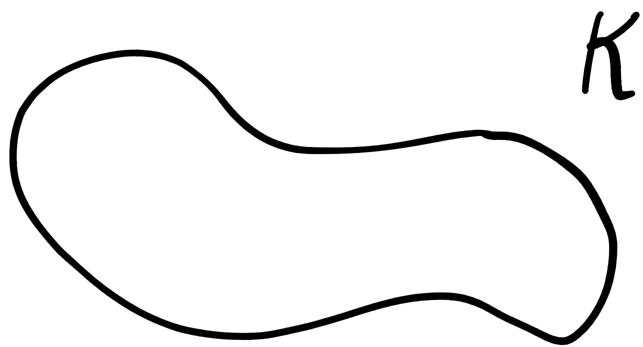
Since compactness is difficult to check, we would like a simpler characterization of compact sets.

Recall:

- Compact $\Rightarrow$ closed, bounded
- Compact $\Leftarrow$ finite

Let (X, d) be a metric space.
A set $K \subseteq X$ is <u>totally bounded</u> if for every $\varepsilon > 0$, there are finitely many points $x_1, \ldots, x_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} B(x_i; \varepsilon)$$



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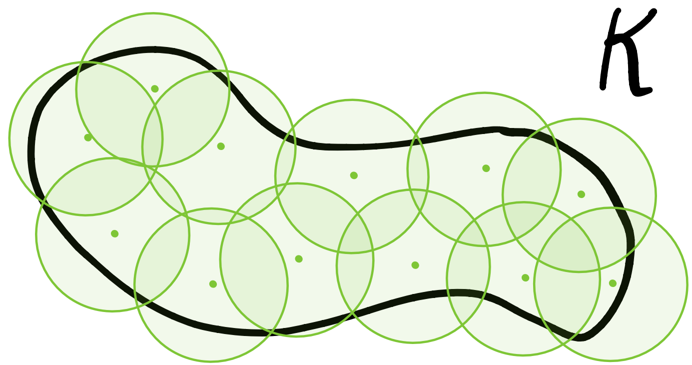
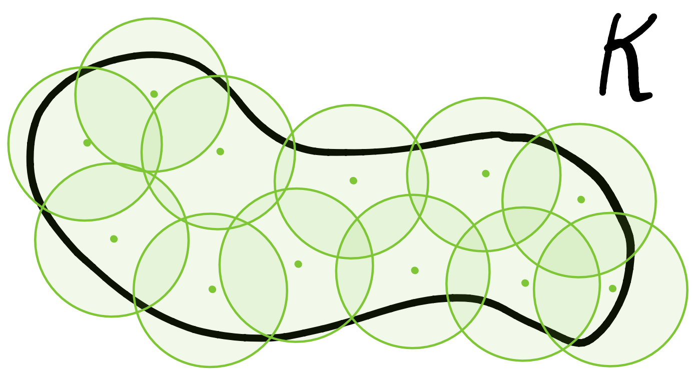


Every totally bounded set is bounded



Every compact set is totally bounded

Proof: Assume $K \subseteq X$ is compact but not totally bounded.

Then $\exists\ \varepsilon > 0$ such that $K \not\subseteq \bigcup_{i=1}^{n} B(x_i; \varepsilon)$ for any choice of $x_1, \dots, x_n \in K$.

Pick any $x_1 \in K$. Then $K \not\subseteq B(x_1; \varepsilon)$, so there is some $x_2 \in K \setminus B(x_1; \varepsilon)$. Iteratively, given $x_1, \dots, x_n$, pick any $x_{n+1} \in K \setminus \bigcup_{i=1}^{n} B(x_i; \varepsilon)$. Note that $d(x_n, x_m) \geq \varepsilon$ for any $n \neq m$.

Then $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in K, so it has a convergent subsequence, $x_{n(k)} \xrightarrow[k \to \infty]{} x \in K$. But then

$$\varepsilon \leq d(x_{n(k)}, x_{n(l)}) \leq d(x_{n(k)}, x) + d(x, x_{n(l)}) \xrightarrow[k, l \to \infty]{} 0 \quad ⚡$$

$\blacksquare$

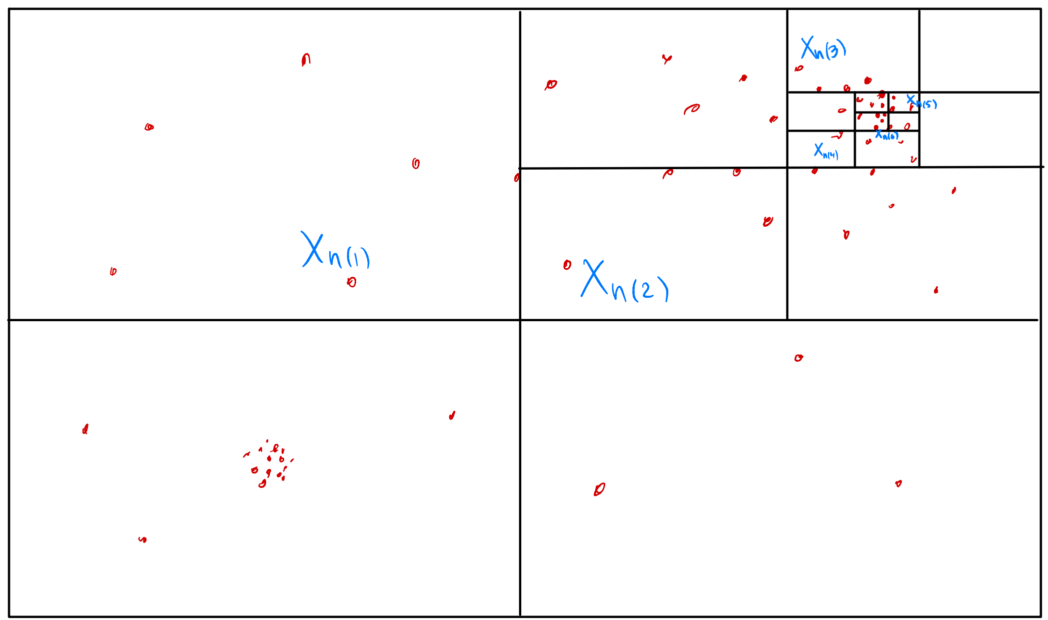
Let (X,d) be complete and $K \subseteq X$. Then

K is compact $\iff$ K is closed and totally bounded.

<u>Proof</u>: We will use the same idea as in the Bolzano-Weierstrass theorem.

The Bolzano-Weierstrass Theorem

If $\{x_n\}_n$ is a bounded sequence in $\mathbb{R}^m$ then it has a convergent subsequence: $x_{n(k)} \xrightarrow[k\to\infty]{} x$,

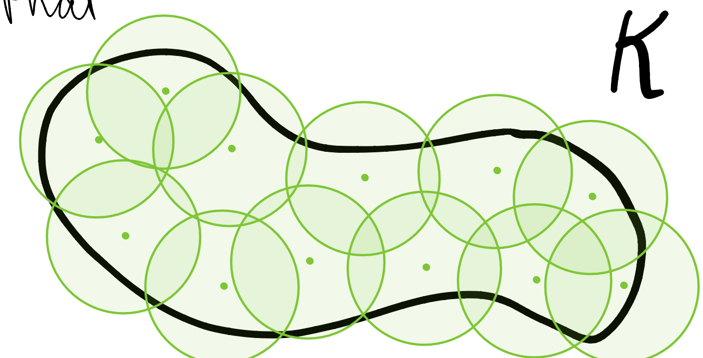


Let (X,d) be complete and $K \subseteq X$. Then

K is compact $\Longleftrightarrow$ K is closed and totally bounded.

"$\Longleftarrow$": Let $\{x_n\}_n$ be a sequence in K. Letting $\varepsilon = 1/2$, there are $y_1, \dots, y_m \in K$ such that

$$K \subseteq \bigcup_{i=1}^{m} B(y_i; 1/2)$$



At least one of the balls $B(y_1; 1/2), \dots, B(y_m; 1/2)$ contains infinitely many of the elements $x_1, x_2, \dots$. Let $n_1(1), n_1(2), n_1(3), \dots \in \mathbb{N}$ be the indices of those elements. Note: $d(x_{n_1(i)}, x_{n_1(j)}) < 1 \;\; \forall i,j$.

Next, letting $\varepsilon = 1/4$, there are $z_1, \ldots, z_{\bar{m}} \in K$ such that $K \subseteq \bigcup_{i=1}^{\bar{m}} B(z_i; 1/4)$. At least one of the balls $B(z_1; 1/4), \ldots, B(z_{\bar{m}}; 1/4)$ contains infinitely many of the elements $x_{n_1(1)}, x_{n_1(2)}, \ldots$. Let $n_2(1), n_2(2), \ldots$ be the indices of those elements. Note: $d(x_{n_2(i)}, x_{n_2(j)}) < \frac{1}{2} \;\forall\, i,j$.

In this way, we pick out sub-sub-sub-....-subsequences satisfying $d\left(x_{n_k(i)}, x_{n_k(j)}\right) < \frac{1}{2^k}$ $\forall i,j$ and hence also

$$d\left(x_{n_k(i)}, x_{n_\ell(j)}\right) < \frac{1}{2^{\min(k,\ell)}} \quad \forall i,j \quad \forall k,\ell$$

$$\begin{array}{cccc}
1 & 2 & 3 & \cdots \\
n_1(1) & n_1(2) & n_1(3) & \cdots \\
n_2(1) & n_2(2) & n_2(3) & \cdots \\
n_3(1) & n_3(2) & n_3(3) & \cdots \\
\vdots & \vdots & \vdots &
\end{array}$$

Now let $n(k) = n_k(k)$.

Now let $n(k) = n_k(k)$. Then $d(x_{n(k)}, x_{n(\ell)}) < \frac{1}{2^N}$ $\forall\, k, \ell \geq N$

so $\{x_{n(k)}\}_{k \in \mathbb{N}}$ is Cauchy! Hence, $x_{n(k)} \xrightarrow[k \to \infty]{} x \in X$.

Since $\{x_{n(k)}\}_{k \in \mathbb{N}}$ lies in K, and K is closed, we have $x \in K$.

It follows that K is compact.

$\blacksquare$

Separability

A <u>separable</u> space is "almost countable".

Compact sets are separable.

Recall: A set D is **countable** if we can make a list of its contents $D = \{d_1, d_2, d_3, \ldots\}$

Let (X, d) be a metric space and let $A \subseteq B \subseteq X$.
The set A is **dense** in B if for all $x \in B$ and $\varepsilon > 0$ there is some $y \in A$ such that $d(x, y) < \varepsilon$.

Examples:

- Any set is dense in itself
- $\mathbb{Q}$ is dense in $\mathbb{R}$
- $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$
- $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$

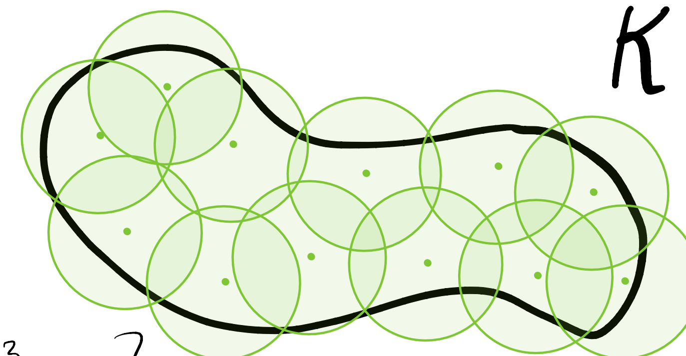
Let (X, d) be a metric space. A set $B \subseteq X$ is <u>separable</u> if there exists a dense, countable set $D \subseteq B$

<u>Examples</u>:

- $\mathbb{R}$ is separable
- Any subset of $\mathbb{R}$ is separable
- If $B_1, B_2, \ldots$ are separable, then so is $\bigcup_{i=1}^{\infty} B_i$
- $\ell^p(\mathbb{R})$ is separable for all $p < \infty$, but <u>not</u> for $p = \infty$.

Every compact set is separable.

<u>Proof</u>: Let $K \subseteq X$ be compact. Then it is totally bounded, so for any $n \in \mathbb{N}$ there are N_n points $x_1^n, \ldots, x_{N_n}^n \in K$ so that $K \subseteq \bigcup_{i=1}^{N_n} B(x_i^n; \frac{1}{n})$.



Let $D = \{x_1^1, \ldots, x_{N_1}^1, x_1^2, \ldots, x_{N_2}^2, x_1^3, \ldots\}$.

Then D is countable, and if $x \in K$, $\varepsilon > 0$ then letting $n > \frac{1}{\varepsilon}$ there is some $x_i^n \in D$ with $d(x, x_i^n) < \frac{1}{n} < \varepsilon$.

$\blacksquare$

QUESTIONS?
COMMENTS?