COMPACTNESS

IV

(BONUS!)

A metric space (X, d) is <u>compact</u> if every sequence $\{x_n\}_n$ has a convergent subsequence $\{x_{n(k)}\}_k$.

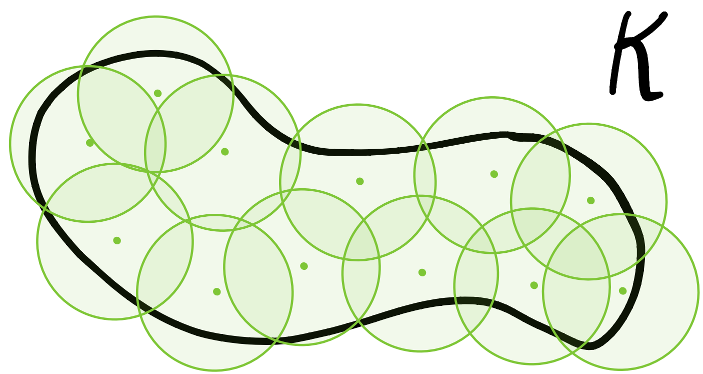
Let (X, d) be complete and $K \subseteq X$. Then

$$K \text{ is compact} \iff K \text{ is closed and totally bounded.}$$

Let (X, d) be a metric space.

A set $K \subseteq X$ is totally bounded if for every $\varepsilon > 0$, there are finitely many points $x_1, \dots, x_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} B(x_i; \varepsilon)$$



Open coverings

Let (X,d) be a metric space and let $K \subseteq X$.

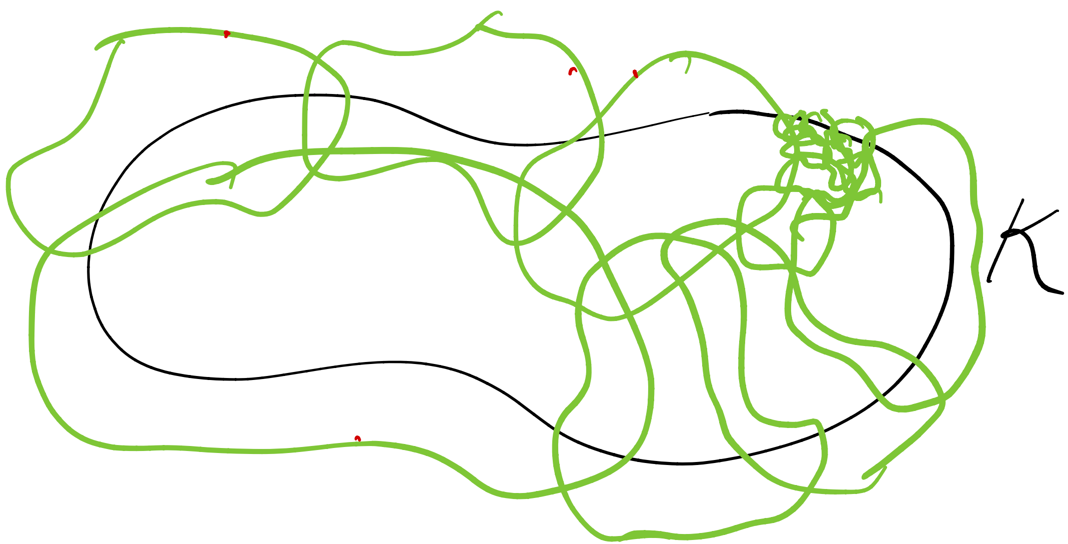
An <u>open covering of K</u> is a family of open sets $\mathcal{F}$ satisfying

$$K \subseteq \bigcup_{O \in \mathcal{F}} O$$

<u>Examples</u>:

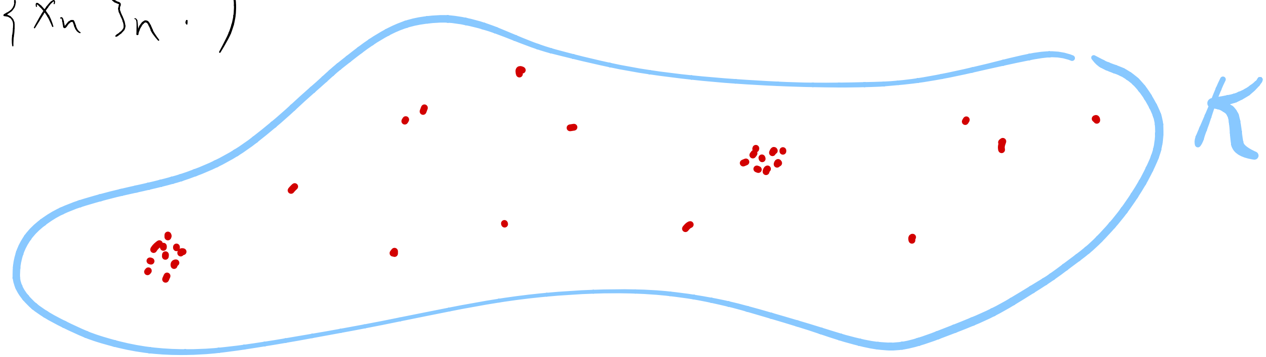
- $\{X\}$ is an open covering of any $K \subseteq X$
- $\left\{ \left(\frac{1}{3n}, \frac{1}{n} \right) : n \in \mathbb{N} \right\}$ is an open covering of $(0,1)$
- $\{ B(x; \varepsilon) : x \in K \}$ is an open covering of K, for any $\varepsilon > 0$.

A set $K \subseteq X$ has the <u>open covering property</u> if for any open covering $\mathcal{F}$ of K, there exist $O_1, \dots, O_n \in \mathcal{F}$ such that $\{O_1, \dots, O_n\}$ is an open covering of K.



Theorem: A set $K \subseteq X$ is compact iff it has the open covering property.

Proof of "$\Longleftarrow$": If $\{x_n\}_n$ is a sequence with a convergent subsequence, then there must be a point $x \in K$ such that $B(x; r)$ contains infinitely many elements of $\{x_n\}_n$, no matter what $r > 0$ is. (x is a cluster point for $\{x_n\}_n$.)



Assume K is <u>not</u> compact. Then there is a sequence $\{x_n\}_n$ in K without convergent subsequences. In particular, <u>no</u> point $x \in K$ is a cluster point for $\{x_n\}_n$.

Thus, for every $x \in K$ there is some $r_x > 0$ such that $B(x; r_x)$ contains only finitely many of $\{x_n\}_n$.

The family $\{B(x; r_x) : x \in K\}$ is an open covering of K, so there are $x_1, \dots, x_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} B(x_i; r_{x_i}).$$

But each $B(x_i; r_{x_i})$, hence also $\bigcup_{i=1}^{n} B(x_i; r_{x_i})$ contains only finitely many elements of $\{x_n\}_n$ $\lightning$ $\blacksquare$

For the converse, we need:

Lemma: Let $K \subseteq X$ be a subset and let $\mathcal{F}$ be an open covering of K. Let

$$f(x) = \sup\{r > 0 : B(x;r) \subseteq O \text{ for some } O \in \mathcal{F}\}.$$

Then f is continuous and $f(x) > 0$ $\forall x \in K$.

Proof: For every $x \in K$ there is some $O \in \mathcal{F}$ with $x \in O$, and O is open, so $B(x;r) \subseteq O$ for some $r > 0$.

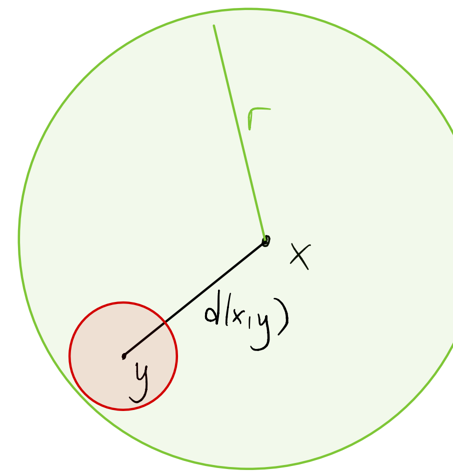
Hence, $f(x) \geq r > 0$.

We claim that f is Lipschitz continuous:

$$|f(x) - f(y)| \leq d(x,y) \qquad \forall\ x, y \in K.$$

If both $f(x)$ and $f(y)$ are $\leq d(x,y)$ then $|f(x)-f(y)| \leq d(x,y)$.

If, say, $f(x)$ is bigger than both $f(y)$ and $d(x,y)$ then for any $r>0$ with $d(x,y) < r < f(x)$ there is some $O \in \mathcal{T}$ with $B(x;r) \subseteq O$.



Then also $B(y; r-d(x,y)) \subseteq B(x;r) \subseteq O$, so

$$f(y) \geq r - d(x,y).$$

Since this is true for any $r < f(x)$, we can let $r \to f(x)$ to get

$$f(y) \geq f(x) - d(x,y) \implies |f(x) - f(y)| \leq d(x,y).$$

$\blacksquare$

Theorem: A set $K \subseteq X$ is compact <u>iff</u> it has the open covering property.

<u>Proof of "$\Rightarrow$"</u>: Let $\mathcal{O}$ be an open covering of K and let $f(x) = \sup\{r > 0 : B(x; r) \subseteq O$ for some $O \in \mathcal{O}\}$.

By the extremal value theorem, there is some $\bar{x} \in K$ where

$$0 < f(\bar{x}) \leq f(x) \quad \forall \ x \in K.$$

Let $r = f(\bar{x})/2$. For every $x \in K$, there is some $O \in \mathcal{O}$ such that $B(x; r) \subseteq O$.

Since K is totally bounded, there are $x_1, \dots, x_n \in K$ s.t.

$$K \subseteq \bigcup_{i=1}^{n} B(x_i; r).$$

If $O_i \in \mathcal{O}$ contains x_i then

$$K \subseteq \bigcup_{i=1}^{n} B(x_i; r) \subseteq \bigcup_{i=1}^{n} O_i.$$

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QUESTIONS?

COMMENTS?