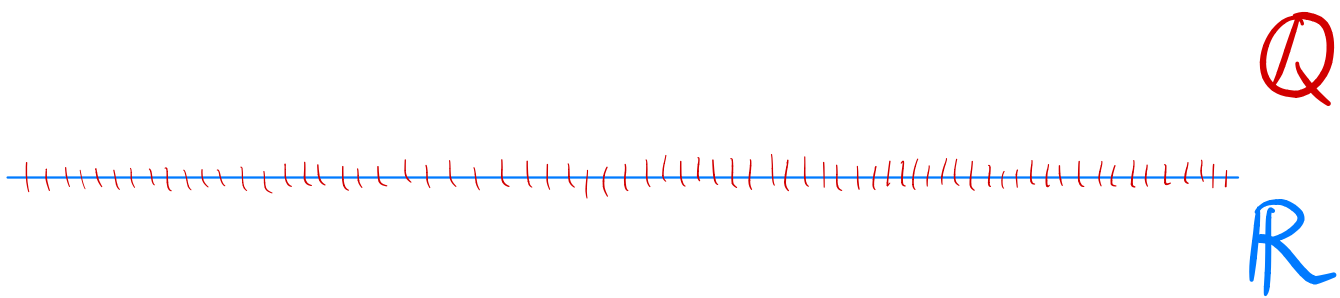
COMPLETION OF A METRIC SPACE

Let (X, d) be an incomplete metric space.
Can we complete (X, d) – add elements to make it complete?

Example: $\mathbb{R}$ is a completion of $\mathbb{Q}$



$$(X, d) \xrightarrow{\text{"add elements"}} (\overline{X}, \overline{d})$$

__Wishlist:__

- $(\overline{X}, \overline{d})$ is complete
- $X \subseteq \overline{X}$ (X can be __embedded__ in $\overline{X}$)
- $d(x,y) = \overline{d}(x,y) \quad \forall\, x,y \in X$ (d and $\overline{d}$ are compatible)
- X is dense in $\overline{X}$ (we are not adding too many elements)

Such a metric space $(\overline{X}, \overline{d})$ is a __completion__ of (X, d)

Completions are unique (if they exist) - up to isometries.

An isometry is a "relabeling" of elements which leaves the metric intact.

For instance, the following mean the same:

- $\sqrt{2}$
- the square root of 2
- the positive solution of $x^2 = 2$

Main idea: Identify each $x \in X$ with all sequences $\{x_n\}_n$ in X such that $x_n \xrightarrow[n\to\infty]{} x$

- Given $\{x_n\}_n$, we can find $x = \lim_{n\to\infty} x_n$
- Given x, we can find all sequences $\{x_n\}_n$ s.t. $x_n \xrightarrow[n\to\infty]{} x$.

Thus:

$$x \longleftrightarrow \bar{x} = \left\{ \{x_n\}_n \text{ in } X \;:\; x_n \xrightarrow[n\to\infty]{} x \right\}.$$

Note: If $\{x_n\}_n \in \bar{x}$ then $\{y_n\}_n \in \bar{x}$ *iff* $d(x_n, y_n) \xrightarrow[n\to\infty]{} 0$.

$$X \longleftrightarrow \overline{X} = \{ \{x_n\}_n \text{ in } X : x_n \xrightarrow[n\to\infty]{} x \}.$$

<u>Note</u>: If $\{x_n\}_n \in \overline{X}$ then $\{y_n\}_n \in \overline{X}$ iff $d(x_n, y_n) \xrightarrow[n\to\infty]{} 0$

Define $[\{x_n\}_n] = \{ \{y_n\}_n \text{ in } X : d(x_n, y_n) \xrightarrow[n\to\infty]{} 0 \}$

("the equivalence class" of $\{x_n\}_n$)

<u>But</u> there are lots of sequences in X that <u>seem to</u> converge (are Cauchy), but don't (as (X,d) is incomplete).

We can still define their equivalence class $[\{x_n\}_n]$!

Let $\overline{X} = \{ [\{x_n\}_n] : \{x_n\}_n$ is a Cauchy sequence in $X \}$

$$\overline{d}([\{x_n\}_n], [\{y_n\}_n]) = \lim_{n\to\infty} d(x_n, y_n).$$

$\boxed{(\overline{X}, \overline{d}) \text{ is a complete metric space}}$

$\boxed{\overline{d}(\overline{x}, \overline{y}) = d(x,y) \quad \forall\ x,y \in X}$

Proof: If $x_n \xrightarrow[n]{} x$, $y_n \xrightarrow[n]{} y$ then $\overline{d}(\overline{x}, \overline{y}) = \lim_n d(x_n, y_n) = d(x,y)$. ▨

$\boxed{(X, d) \text{ is dense in } (\overline{X}, \overline{d})}$

Proof: Given $[\{x_n\}_n] \in \overline{X}$, let $x = x_N$ $(N \in \mathbb{N})$. Then

$$\overline{d}([\{x_n\}_n], \overline{x}) = \lim_{n\to\infty} d(x_n, x_N) \xrightarrow[N\to\infty]{} 0.$$ ▨

Every metric space has a completion.
The completion is unique (up to isometry)

But what does this mean?

- If $X = \mathbb{Q}$ then $\overline{X}$ is a set of sets of sequences, not the set of numbers $\mathbb{R}$!
- In the end, it does not matter what a mathematical object "is", only <u>what we can do with it</u>.
- In $\overline{\mathbb{Q}}$ we can define exactly the same operations as in $\mathbb{R}$.

QUESTIONS ?

COMMENTS ?