MAT2400: Solution to Mandatory Assignment 1, Spring 2022

After the solutions I have added a text in blue where I try to explain how one might think in order to find the right approach. Consider the text in black as the precise and polished version of the text in blue.

Problem 1 a) If all the elements in the sequence are zero, there is nothing to prove, so we may assume that there is an element x_k such that $\epsilon := |x_k|$ is positive. Since $\lim_{n\to\infty} x_n = 0$, there is an $N \in \mathbb{N}$ such that $|x_n| < \epsilon$ for all $n \geq N$. This means that we can choose x_K to be the element with the largest absolute value among $|x_1|, |x_2|, \dots, |x_N|$, as no later element can compete.

Before we begin, it's a good idea to understand what we are asked for. The problem doesn't only ask us to show that a sequence $\{x_n\} \in X$ is bounded, but also that the supremum is *attained*, i.e. that there exists a term x_K such that $|x_K| = \sup\{|x_n| : n \in \mathbb{N}\}$. This isn't the same thing for all sequences: The sequence $\{-\frac{1}{n}\}$ is bounded, but there is no element in the sequence that attains the supremum 0. Fortunately, this is not a counterexample to our problem as we are only dealing with nonnegative sequences, but it shows us that it's important to use the fact that the elements are nonnegative.

We are now ready to get started. If all the terms are 0, there is nothing to prove, so we can concentrate on the case where at least one term $|x_k|$ is nonzero. Since the sequence converges to 0, the terms will eventually become smaller than $|x_k|$. This means that the largest values have to come early in the sequence, and since we then only have finitely many candidates to choose among, one of them must be the largest.

b) Since $\{|x_n - y_n|\}_{n\in\mathbb{N}}$ is an element in X, part a) tells us that $\sup\{|x_n - y_n| : n \in \mathbb{N}\}$ is finite, and hence d is well-defined. We need to check the three defining properties of a metric:

1. *Positivity:* Clearly $d(\{x_n\}, \{y_n\}) = \sup\{|x_n - y_n| : n \in \mathbb{N}\} \geq 0$ as it's a supremum of nonnegative numbers, and just as clearly $d(\{x_n\}, \{x_n\}) = 0$. On the other hand, if $\{x_n\} \neq \{y_n\}$, there is an n such that $|x_n - y_n| > 0$, and hence $d(\{x_n\}, \{y_n\}) > 0$.

2. *Symmetry:* Since $|x_n - y_n| = |y_n - x_n|$, this follows directly from the definition.

3. *Triangle inequality:* Assume that $\{x_n\}, \{y_n\}, \{z_n\} \in X$. As the sequence $\{|x_n - y_n|\}_{n\in\mathbb{N}}$ is an element of X, it follows from a) that there is a K such that

$$|x_K - y_K| = \sup\{|x_n - y_n| : n \in \mathbb{N}\} = d(\{x_n\}, \{y_n\}),$$

But then

$$d(\{x_n\},\{y_n\}) = |x_K - y_K| \le |x_K - z_K| + |z_K - y_K|$$
$$\le d(\{x_n\},\{z_n\}) + d(\{z_n\},\{y_n\}),$$

where we have used the triangle inequality in $\mathbb{R}$.

The first two parts should be fairly obvious. To get the triangle inequality for sequences, it seems natural to use the triangle inequality for numbers (it's basically the only tool we have), and it's then natural to start with $|x_k - y_k|$ for some k. It is not necessary to use the K from part a), but it simplifies things a little.

c) If $\{y_n\} \in Y$, the series $\sum_{n=1}^{\infty} |y_n|$ converges, and hence the sequence $\{y_n\}$ of terms converges to 0 (this is well-known from calculus and easy to prove, see, e.g. *Kalkulus* 12.1.4) and belongs to X. We also know from calculus that the *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges despite the terms $\frac{1}{n}$ going to 0. Hence $\{\frac{1}{n}\} \in X \setminus Y$.

This problem is basically intended to get you in the right state of mind before d) and e) by reminding you of a few facts from MAT1110.

d) As $\{x_n\} \in X \setminus Y$, the series $\sum_{n=1}^{\infty} |x_n|$ diverges, but the terms go to 0, and hence there is an $N \in \mathbb{N}$ such that $|x_n| < \epsilon$ when $n \ge N$. Put

$$\hat{x}_n = \begin{cases} x_n & \text{if } n < N \\ 0 & \text{if } n \ge N \end{cases}$$

Then $d(\{x_n\},\{\hat{x}_n\}) < \epsilon$, but $\sum_{n=1}^{\infty} |\hat{x}_n|$ clearly converges, and hence $\{\hat{x}_n\} \in Y$. This shows that any ball around $\{x_n\} \in X \setminus Y$ contains elements from Y. Thus $X \setminus Y$ cannot be open, and consequently Y is not closed.

Let us start by rephrasing the problem in more elementary terms: We start with a sequence $\{x_n\}$ such that $x_n \to 0$, but $\sum_{n=1}^{\infty} |x_n|$ diverges, and want to modify it to get a sequence $\{\hat{x}_n\}$ such that $\sum_{n=1}^{\infty} |\hat{x}_n|$ converges, but $\{\hat{x}_n\}$ is still closer to $\{x_n\}$ than ϵ. The easiest way to find a sequence $\{\hat{x}_n\}$ close to $\{x_n\}$, is to have it replicate $\{x_n\}$ until x_n becomes small. The easiest way to make a series converge, is to make all terms equal to 0 from some term onwards. When we have realized this, the definition of $\hat{x}_n$ becomes quite natural.

e) As $\{y_n\} \in Y$, the series $\sum_{n=1}^{\infty} |y_n|$ converges. Let $N \in \mathbb{N}$ be so large that $\frac{1}{N} < \epsilon$, and define a new sequence $\{\hat{y}_n\}$ by

$$\hat{y}_n = \begin{cases} y_n & \text{if } n < N \\ y_n + \frac{c_n}{n} & \text{if } n \ge N \end{cases}$$

where c_n is 1 or -1 according to whether y_n is positive or negative (and hence $|\hat{y}_n| = |y_n| + \frac{1}{n}$ for $n \geq N$). Then $d(\{y_n\}, \{\hat{y}_n\}) < \epsilon$, but $\sum_{n=1}^{\infty} |\hat{y}_n| = \sum_{n=1}^{\infty} |y_n| + \sum_{n=N}^{\infty} \frac{1}{n}$ diverges as it is the sum of a convergent and a divergent series. Hence $\{\hat{y}_n\}$ is in $B(\{y_n\}; \epsilon)$, but not in Y. This means that all balls around $\{y_n\} \in Y$ contain points that are not in Y, and hence Y is not open.

Let us start by rephrasing the problem in more elementary terms as we did with d): This time we start with a sequence $\{y_n\}$ such that $y_n \to 0$ and $\sum_{n=1}^{\infty} |y_n|$ converges, and want to modify it to get a sequence $\{\hat{y}_n\}$ such that $\hat{y}_n \to 0$ and $\sum_{n=1}^{\infty} |\hat{y}_n|$ diverges, but $\{\hat{y}_n\}$ is still closer to $\{y_n\}$ than ϵ. As above, the easiest way to find a sequence $\{\hat{y}_n\}$ close to $\{y_n\}$, is to have it replicate $\{y_n\}$ until y_n becomes small. The easiest way to change a convergent series into a divergent one, is to add a divergent series to it, but to make the sequences close, we have to wait till the terms of the divergent series become small. The c_n's are just a trick to make the argument smoother.

Problem 2 a) Put $O_1 = [0, 1]$, $O_2 = [2, 3]$. Clearly $O_1 \cup O_2 = X$ and $O_1 \cap O_2 = \emptyset$, and we only need to prove that O_1 and O_2 are open in X (although they are not open in $\mathbb{R}$, they can still be open in X). Observe that if $a \in O_1$, then the ball

$$B(a; \frac{1}{2}) = \{x \in X \colon |x - a| < \frac{1}{2}\}$$

is contained in O_1, and thus a is an interior point of O_1. This means that *all* elements in O_1 are interior, and hence O_1 is open. A similar argument shows that O_2 is open.

As X is the disjoint union of $[0, 1]$ and $[2, 3]$, it is tempting to think that we can put $O_1 = [0, 1]$ and $O_2 = [2, 3]$. The problem is that these sets seem to be closed and not open, but it is worth checking if they can be open, too. (There is actually an easier, but less informative way to show that $[0, 1]$ and $[2, 3]$ are open: Just check that their complements $[0, 1]^c = [2, 3]$ and $[2, 3]^c = [0, 1]$ are closed in X.)

b) Clearly, $O_1 \cap O_2 = \emptyset$, and since there is no rational number x such that $x^2 = 2$, we must have $Q_1 \cup O_2 = \mathbb{Q}$. It remains to prove that O_1 and O_2 are open in $\mathbb{Q}$. Assume that $a \in O_1$. Then $a^2 > 2$, and we need to prove that all x in a neighborhood of a also belong to O_1, i.e. that $x^2 > 2$. There are several ways to do this, but the quickest may be to observe that since the function $f(x) = x^2$ is continuous and $f(a) > 2$, we must also have $f(x) > 2$ for all x in a neighborhood of a. The proof that O_2 is open is similar (just replace $>$ by $<$).

The solution is more or less given by the hint, but there is a little bit of checking to do. There are more elementary ways to do this than the trick with $f(x) = x^2$.

c) Assume for contradiction that there is no c such that $f(c) = 0$. Put $O_1 = \{x \in X : f(x) > 0\} = f^{-1}((0, \infty))$ and $O_2 = \{x \in X : f(x) < 0\} = f^{-1}((-\infty, 0))$. Then O_1 and O_2 are open (they are inverse images of open sets under the continuous function f), $O_1 \cap O_2 = \emptyset$, and (since there is no point c where $f(c) = 0$) $O_1 \cup O_2 = X$. Hence X is disconnected, which is a contradiction as X is assumed to be connected.

As the only thing we have to work with, is the very brief definition of a disconnected set, it is natural to try a proof by contradiction (or a contrapositive proof) to bring this definition into play. This means that we are looking for two disjoint, open sets that make up the entire space, and the crucial observation is that if there is no point c where $f(c) = 0$, then X splits naturally into two parts: $O_1 = \{x \in X : f(x) > 0\}$ and $O_2 = \{x \in X : f(x) < 0\}$. As we have strict inequalities, these sets have an open look about them, and we just need to check that they really are open. Since the statement does not hold for discontinuous f, we need to use the continuity of f somewhere in the argument.