9th February, 2023

MAT2400

Mandatory assignment 1 of 2

Submission deadline

Thursday 23rd FEBRUARY 2023, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Each question counts for 10 points. The requirement to pass the mandatory assignment is to score at least 50 points, out of 90. The two first exercises will be given a pass/fail mark (10 or 0 points) based on reasonable effort.

Exercise 1 (reflection diary)**.** As indicated on the course pages, the compulsory exercise has a diary component. Write four sentences of reflection per week, including at least one sentence on what you found interesting and one sentence on what you found challenging. The diary should cover the four first weeks of lectures.

Exercise 2 (self evaluation)**.** After you have handed in the assignment you will be given suggestions for solutions. With the help of these you will be asked to correct your own assignment and hand it in again. Your self evaluation will be corrected by the teaching assistant.

Exercise 3. Notation: $\mathbb{R}_+^* = \{x \in \mathbb{R} \ : \ x > 0\}$, $\mathbb{N}^* = \{n \in \mathbb{N} \ : \ n > 0\}$.

(i) Let $x \in \mathbb{R}$. Suppose that for all $\epsilon \in \mathbb{R}_+^*$ we have $x < \epsilon$. Argue by contradiction to show that $x \leq 0$.

> Suppose on the contrary that $x > 0$. We let $\epsilon = x/2$. We get $x < x/2$, so $x/2 < 0$, hence $x < 0$. This is a contradiction.

(ii) Let $x \in \mathbb{R}$. Suppose that there exists $C \in \mathbb{R}_+^*$ such that for all $n \in \mathbb{N}^*$ we have $x \leq C/n$. Use Archimedes principle to show that $x \leq 0$.

> Let $C \in \mathbb{R}_+^*$ be such that for all $n \in \mathbb{N}^*$ we have $x \leq C/n$. If we don't have $x \leq 0$ we get $x > 0$. Archimedes principle provides an $N \in \mathbb{N}^*$ such that $Nx > C$, that is $x > C/N$. This contradicts the defining property of C.

Exercise 4. (i) Suppose that $u :]0,1] \to \mathbb{R}$ is a bounded function. Define a function $v : [0,1] \to \mathbb{R}$ by $v(0) = 0$ and, for all $x \in]0,1]$, $v(x) = x^2 u(x)$. Show that v is differentiable in 0.

> Choose $C > 0$ such that for all $x \in]0,1]$ we have $|u(x)| \leq C$. For $x \in]0,1]$ we have:
>
> $$|\frac{v(x) - v(0)}{x - 0}| = |\frac{v(x)}{x}| = |xu(x)| \leq Cx. \tag{1}$$
>
> The limit of this expression as $x \to 0$ is 0, so v is differentiable in 0.

(ii) Find a function $v : [0,1] \to \mathbb{R}$ such that v is differentiable on $[0,1]$ and v' is unbounded on $[0,1]$. You may want to apply the preceding construction to a rapidly oscillating function u.

We define $u :]0,1] \to \mathbb{R}$, by:

$$u(x) = \sin(1/x^2). \tag{2}$$

We define v as in the previous question. Then v is differentiable on $[0,1]$. For $x \in]0,1]$ we have:

$$\begin{aligned} v'(x) &= x^2 \cos(1/x^2)(-2/x^3) + 2x \sin(1/x^2), && (3) \\ &= 2(-\cos(1/x^2)/x + x \sin(1/x^2)). && (4) \end{aligned}$$

Define $x_n = \sqrt{(1/(2\pi n)}$ for $n \in \mathbb{N}^*$. Then $v'(x_n) = -2\sqrt{(2\pi n)} \to -\infty$. Hence v' is unbounded.

Exercise 5. (i) Let $a_0, b_0 \in \mathbb{R}$ be such that $a_0 < b_0$. Let $x_1 \in \mathbb{R}$ also be given. Show that there exist $a_1, b_1 \in \mathbb{R}$ such that $a_0 \leq a_1 < b_1 \leq b_0$ and $(b_1 - a_1) = (1/3)(b_0 - a_0)$, and also $x_1 \notin [a_1, b_1]$. You may want to discuss according to the position of x_1 relative to a_0 and b_0.

Suppose first $x_1 > a_0 + (1/3)(b_0 - a_0)$. Then we put $a_1 = a_0$ and $b_1 = a_0 + (1/3)(b_0 - a_0)$.
If not we have $x_1 \leq a_0 + (1/3)(b_0 - a_0) < b_0 - (1/3)(b_0 - a_0)$. Then we put $a_1 = b_0 - (1/3)(b_0 - a_0)$ and $b_1 = b_0$.

(ii) Suppose (x_n) is a sequence in $\mathbb{R}$. Show that there exist sequences[1] (a_n) and (b_n) in $\mathbb{R}$ such that (a_n) is increasing, (b_n) is decreasing , $a_n < b_n$ for all $n \in \mathbb{N}$, $(b_n - a_n)$ converges to zero, and moreover $x_n \notin [a_n, b_n]$.

First find a_0, b_0 such that $a_0 < b_0$ and $x_0 \notin [a_0, b_0]$.
Supposing we have found adequate $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$, we use the preceding question to define a_{n+1} and b_{n+1} such that $a_{n+1} \geq a_n$ and $b_{n+1} \leq b_n$ and $b_{n+1} - a_{n+1} = (1/3)(b_n - a_n)$ and $x_{n+1} \notin [a_{n+1}, b_{n+1}]$.
We get sequences such that $(b_n - a_n) = (1/3^n)(b_0 - a_0)$ and this converges to 0.

(iii) Use the preceding question to show that $\mathbb{R}$ is not countable.

[1] Such sequences are called adjacent.

Suppose, for contradiction, that $\mathbb{R}$ is countable. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ which is also a surjection $\mathbb{N} \to \mathbb{R}$. Construct (a_n) and (b_n) as in the preceding question. Let $u \in \mathbb{R}$ be a common limit of those two sequences (here we use the completeness of $\mathbb{R}$!). We have that $u \in [a_n, b_n]$ for each $n \in \mathbb{N}$. In particular $u \neq x_n$. This contradicts the surjectivity of the sequence (x_n).