11th April, 2023

MAT2400

Mandatory assignment 2 of 2

Submission deadline

Thursday 27th APRIL 2023, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

This set consists of 9 questions counting 10 points each. The two questions in exercise 2 are optional. The requirement to pass is to score 40 points on the remaining 7 questions.

Exercise 1 (reflection diary)**.** As indicated on the course pages, the compulsory exercise has a diary component. Write four sentences of reflection per week, covering the last three weeks preceding the date for handing in this assignment. Also write four sentences of synthesis about the semester so far.

Exercise 2 (Optional)**.** (i) Suppose $v : [0, T] \to \mathbb{R}$ is a function of class C^1. Suppose we have $A, B \in \mathbb{R}$ with $B > 0$ such that for all $t \in [0, T]$:

$$v'(t) \leq A + Bv(t). \tag{1}$$

Multiply by an integrating factor to deduce that:

$$v(t) \leq v(0)e^{Bt} + A\frac{e^{Bt} - 1}{B}. \tag{2}$$

(ii) (Grönwall's inequality) Suppose $u : [0, T] \to \mathbb{R}$ is a continuous function. Suppose we have $A, B \in \mathbb{R}$ with $B > 0$ such that for all $t \in [0, T]$:

$$u(t) \leq A + B \int_0^t u(s)\mathrm{d}s. \tag{3}$$

Use the preceding question to deduce that, for all $t \in [0, T]$:

$$u(t) \leq Ae^{Bt}. \tag{4}$$

Exercise 3. In this exercise you are encouraged to use Grönwall's inequality from Exercise 2, even if you haven't proved it.

In practice one might not be able to solve a given differential equation exactly. A numerical method, for instance, only solves the equation up to some tolerance term. This exercise is about controlling the error in the solution, for given tolerances in the differential equation. The game is also to not use the known theorem of existence and uniqueness of solutions to differential equations, but rather to provide a new proof.

Fix a norm on $\mathbb{R}^n$, denoted $\| \cdot \|$. Let $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and Lipschitz in the second variable uniformly in $t \in [0, T]$. That is, there is $M \geq 0$ such that for all $t \in [0, T]$ and all $x, y \in \mathbb{R}^n$ we have:

$$\|f(t, x) - f(t, y)\| \leq M\|x - y\|. \tag{5}$$

Let $x_0 \in \mathbb{R}^n$ also be given. We consider differential equation:

$$\forall t \in [0,T] \quad x'(t) = f(t, x(t)), \tag{6}$$
$$x(0) = x_0. \tag{7}$$

with unknown $x : [0,T] \to \mathbb{R}^n$.

We say that $y : [0,T] \to \mathbb{R}$ is an (μ, ν)-solution of (6, 7) if y is of class C^1 and:

$$\forall t \in [0,T] \quad \|y'(t) - f(t, y(t))\| \leq \mu, \tag{8}$$
$$\|y(0) - x_0\| \leq \nu. \tag{9}$$

(i) Let y and z be two (μ, ν)-solutions of (6, 7). Show that for all $t \in [0,T]$:

$$\|y(t) - z(t)\| \leq (2\nu + 2\mu T)e^{Mt}. \tag{10}$$

(ii) Let (μ_n) and (ν_n) be positive decreasing sequences in $\mathbb{R}$ that converge to 0. Suppose we have a sequence of C^1 functions $y_n : [0,T] \to \mathbb{R}^n$ such that y_n is a (μ_n, ν_n)-solution of (6, 7). Using a completeness argument, show that (y_n) converges uniformly to some continuous function $y : [0,T] \to \mathbb{R}^n$.

(iii) Show that the obtained function y actually is a solution to (6, 7). In view of the integral formulation of differential equations it may be useful to study the convergence of the expression:

$$y_n(t) - y_n(0) - \int_0^t f(s, y_n(s))\mathrm{d}s. \tag{11}$$

Exercise 4. Let $u : \mathbb{R} \to \mathbb{R}$ be a continuous function, which is also periodic. Show that u is uniformly continuous.

Exercise 5. Let X be a metric space. Suppose we have a family $(A_i)_{i \in I}$ such that I is finite and each A_i (for $i \in I$) is a compact subset of X. Show that the set:

$$\bigcup_{i \in I} A_i, \tag{12}$$

is compact.

Exercise 6. We equip $\mathbb{R}^m$ and $\mathbb{R}^n$ with their respective sup-norms, denoted $\| \cdot \|_\infty$. Let $A \in \mathbb{R}^{m \times n}$ be a (nonzero) real matrix. Recall the definition of the operator norm:

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}. \tag{13}$$

Show that we have the equality:

$$\|A\| = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |A_{ij}|. \tag{14}$$

It may be useful to treat the equality as two inequalities that need separate approaches. It may also be useful to look at what happens to vectors x such that each $x_j = \pm 1$ (sign vectors).