8^{th} May, 2023

MAT2400

Mandatory assignment 2 of 2

Submission deadline

Thursday 27^{th} APRIL 2023, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

This set consists of 9 questions counting 10 points each. The two questions in exercise 2 are optional. The requirement to pass is to score 40 points on the remaining 7 questions.

Exercise 1 (reflection diary)**.** As indicated on the course pages, the compulsory exercise has a diary component. Write four sentences of reflection per week, covering the last three weeks preceding the date for handing in this assignment. Also write four sentences of synthesis about the semester so far.

Exercise 2 (Optional)**.** (i) Suppose $v : [0, T] \to \mathbb{R}$ is a function of class C^1. Suppose we have $A, B \in \mathbb{R}$ with $B > 0$ such that for all $t \in [0, T]$:

$$v'(t) \leq A + Bv(t). \tag{1}$$

Multiply by an integrating factor to deduce that:

$$v(t) \leq v(0)e^{Bt} + A\frac{e^{Bt} - 1}{B}. \tag{2}$$

We have:

$$\begin{align} v'(t) - Bv(t) &\leq A, \tag{3}\\ (v'(t) - Bv(t))e^{-Bt} &\leq Ae^{-Bt}, \tag{4}\\ (v(t)e^{-Bt})' &\leq Ae^{-Bt}, \tag{5}\\ v(t)e^{-Bt} - v(0) &\leq A\frac{e^{-Bt} - 1}{-B}, \tag{6}\\ v(t) &\leq v(0)e^{Bt} + A\frac{e^{Bt} - 1}{B}. \tag{7} \end{align}$$

(ii) (Grönwall's inequality) Suppose $u : [0, T] \to \mathbb{R}$ is a continuous function. Suppose we have $A, B \in \mathbb{R}$ with $B > 0$ such that for all $t \in [0, T]$:

$$u(t) \leq A + B\int_0^t u(s)\mathrm{d}s. \tag{8}$$

Use the preceding question to deduce that, for all $t \in [0, T]$:

$$u(t) \leq Ae^{Bt}. \tag{9}$$

Define v by:

$$v(t) = \int_0^t u(s)\mathrm{d}s. \tag{10}$$

We get:

$$v'(t) \leq A + Bv(t). \tag{11}$$

Therefore:

$$\begin{align} u(t) &\leq A + Bv(t), \tag{12}\\ &\leq A + BA\frac{e^{Bt}-1}{B}, \tag{13}\\ &\leq Ae^{Bt}. \tag{14} \end{align}$$

Exercise 3. In this exercise you are encouraged to use Grönwall's inequality from Exercise 2, even if you haven't proved it.

In practice one might not be able to solve a given differential equation exactly. A numerical method, for instance, only solves the equation up to some tolerance term. This exercise is about controlling the error in the solution, for given tolerances in the differential equation. The game is also to not use the known theorem of existence and uniqueness of solutions to differential equations, but rather to provide a new proof.

Fix a norm on $\mathbb{R}^n$, denoted $\|\cdot\|$. Let $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and Lipschitz in the second variable uniformly in $t \in [0,T]$. That is, there is $M \geq 0$ such that for all $t \in [0,T]$ and all $x, y \in \mathbb{R}^n$ we have:

$$\|f(t,x) - f(t,y)\| \leq M\|x - y\|. \tag{15}$$

Let $x_0 \in \mathbb{R}^n$ also be given. We consider differential equation:

$$\begin{align} \forall t \in [0,T] \quad x'(t) &= f(t, x(t)), \tag{16}\\ x(0) &= x_0. \tag{17} \end{align}$$

with unknown $x : [0,T] \to \mathbb{R}^n$.

We say that $y : [0,T] \to \mathbb{R}$ is an (μ,ν)-solution of (16, 17) if y is of class C^1 and:

$$\begin{align} \forall t \in [0,T] \quad \|y'(t) - f(t, y(t))\| &\leq \mu, \tag{18}\\ \|y(0) - x_0\| &\leq \nu. \tag{19} \end{align}$$

(i) Let y and z be two (μ,ν)-solutions of (16, 17). Show that for all $t \in [0,T]$:

$$\|y(t) - z(t)\| \leq (2\nu + 2\mu T)e^{Mt}. \tag{20}$$

We have:

$$
\begin{aligned}
y(t) - z(t) &= y(0) - z(0) + \int_0^t (y'(s) - z'(s))\mathrm{d}s, \\
\|y(t) - z(t)\| &\leq \|y(0) - z(0)\| + \int_0^t \|y'(s) - z'(s)\|\mathrm{d}s, \\
&\leq \|y(0) - z(0)\| + \int_0^t (2\mu + \|f(s, y(s)) - f(s, z(s))\|)\mathrm{d}s, \\
&\leq 2\nu + 2\mu t + M \int_0^t \|y(s) - z(s)\|\mathrm{d}s. \qquad (21)
\end{aligned}
$$

Conclude by Grönwall's inequality.

(ii) Let (μ_n) and (ν_n) be positive decreasing sequences in $\mathbb{R}$ that converge to 0. Suppose we have a sequence of C^1 functions $y_n : [0, T] \to \mathbb{R}^n$ such that y_n is a (μ_n, ν_n)-solution of (16, 17). Using a completeness argument, show that (y_n) converges uniformly to some continuous function $y : [0, T] \to \mathbb{R}^n$.

Consider $n, p, q \in \mathbb{N}$ with $p, q \geq n$. Then y_p and y_q are both (μ_n, ν_n)-solutions of (16). We deduce:

$$
\|y_p(t) - y_q(t)\| \leq (2\mu_n + 2\nu_n T)e^{MT}. \qquad (22)
$$

This bound is uniform in $t \in [0, T]$. It follows that the sequence (y_n) is Cauchy in $\mathrm{C}([0, T], \mathbb{R}^n)$ equipped with the sup-norm. This space is complete. Hence the sequence converges uniformly to some continuous function.

(iii) Show that the obtained function y actually is a solution to (16, 17). In view of the integral formulation of differential equations it may be useful to study the convergence of the expression:

$$
y_n(t) - y_n(0) - \int_0^t f(s, y_n(s))\mathrm{d}s. \qquad (23)
$$

We have:

$$\|y_n(t) - y_n(0) - \int_0^t f(s, y_n(s))\mathrm{d}s\| \tag{24}$$

$$\leq \| \int_0^t (y_n'(s) - f(s, y_n(s)))\mathrm{d}s\|, \tag{25}$$

$$\leq \int_0^t \|y_n'(s) - f(s, y_n(s))\|\mathrm{d}s, \tag{26}$$

$$\leq T\mu_n. \tag{27}$$

Now in (24) we can pass to the limit in the first two terms by simple convergence. Moreover:

$$\| \int_0^t f(s, y_n(s))\mathrm{d}s - \int_0^t f(s, y(s))\mathrm{d}s\| \leq M \int_0^t \|y_n(s) - y(s)\|\mathrm{d}s. \tag{28}$$

Now we can use uniform convergence of y_n to y to control the right hand side.
We can therefore pass to the limit in the third term of (24). We obtain:

$$y(t) = y(0) + \int_0^t f(s, y(s))\mathrm{d}s. \tag{29}$$

Therefore y is a solution to (16, 17).

Exercise 4. Let $u : \mathbb{R} \to \mathbb{R}$ be a continuous function, which is also periodic. Show that u is uniformly continuous.

Let $T > 0$ be a period of u. Now u is continuous on the interval $[-1, T+1]$ which is compact, so u is uniformly continuous on it. Let $\epsilon > 0$. Choose $\delta > 0$ such that for $x, y \in [-1, T+1]$ we have:

$$|x - y| < \delta \implies |u(x) - u(y)| < \epsilon. \tag{30}$$

We may reduce δ to guarantee in addition $\delta < 1$. Suppose now that $x, y \in \mathbb{R}$ and that $|x - y| < \delta$. Choose $n \in \mathbb{Z}$ such that $x' = x - nT \in [0, T]$. Put $y' = y - nT$. Now we have $x', y' \in [-1, T+1]$ and $|x' - y'| = |x - y| < \delta$, so $|u(x') - u(y')| < \epsilon$. Therefore $|u(x) - u(y)| < \epsilon$.

Exercise 5. Let X be a metric space. Suppose we have a family $(A_i)_{i\in I}$ such that I is finite and each A_i (for $i \in I$) is a compact subset of X. Show that the set:

$$\bigcup_{i\in I} A_i, \tag{31}$$

is compact.

Let B be the union of the sets A_i, $i \in I$. Let (u_n) be a sequence in B. For each $i \in I$ consider the set:

$$J_i = \{n \in \mathbb{N} \ : \ u_n \in A_i\}. \tag{32}$$

Since the union of the sets J_i is $\mathbb{N}$, one of them must be infinite, say the one corresponding to index $i_0 \in I$. Let (v_n) be the corresponding subsequence of (u_n). It is a sequence in A_{i_0}. Choose a subsequence (w_n) which converges in A_{i_0}, to say $x \in A_{i_0}$. Then (w_n) is a subsequence of (u_n) that converges to $x \in B$.

Exercise 6. We equip $\mathbb{R}^m$ and $\mathbb{R}^n$ with their respective sup-norms, denoted $\|\cdot\|_\infty$. Let $A \in \mathbb{R}^{m\times n}$ be a (nonzero) real matrix. Recall the definition of the operator norm:

$$|\!|\!|A|\!|\!| = \sup_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}. \tag{33}$$

Show that we have the equality:

$$|\!|\!|A|\!|\!| = \max_{1\leq i\leq m} \sum_{j=1}^{n} |A_{ij}|. \tag{34}$$

It may be useful to treat the equality as two inequalities that need separate approaches. It may also be useful to look at what happens to vectors x such that each $x_j = \pm 1$ (sign vectors).

Let $x \in \mathbb{R}^n$, put $y = Ax$. We have:

$$|y_i| = |\sum_{j=1}^{m} A_{ij} x_j|, \tag{35}$$

$$\leq \sum_{j=1}^{m} |A_{ij}|\,|x_j|, \tag{36}$$

$$\leq \sum_{j=1}^{m} |A_{ij}| \|x\|_\infty. \tag{37}$$

This gives:

$$\|A\| \leq \max_{1 \leq i \leq m} \sum_{j=1}^{n} |A_{ij}|. \tag{38}$$

Now suppose that $\max_{1 \leq i \leq m} \sum_{j=1}^{n} |A_{ij}|$ is attained for index $i = i_0$. Define x by $x_j = \operatorname{sign} A_{ij}$ and y by $y = Ax$.
Then:

$$y_{i_0} = \sum_{j=1}^{m} A_{i_0 j} x_j = \sum_{j=1}^{m} |A_{i_0 j}|. \tag{39}$$

So:

$$\|y\|_\infty \geq \max_{1 \leq i \leq m} \sum_{j=1}^{n} |A_{ij}| \|x\|_\infty. \tag{40}$$

This proves:

$$\|A\| \geq \max_{1 \leq i \leq m} \sum_{j=1}^{n} |A_{ij}|. \tag{41}$$