UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MAT2400 — Real Analysis
Day of examination:	Friday 2. June 2023
Examination hours:	15:00 – 19:00

This problem set consists of 3 pages.

Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

For each question you may use results from the preceding questions even if you haven't answered them.

Problem 1 (weight 20%)

Let (X, d) be a metric space.

a (weight 10%)

Suppose that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Suppose furthermore that for a certain $a \in X$, (u_n) has a subsequence that converges to a. Show that (u_n) converges to a.

b (weight 10%)

In this question any subset of X is considered as a metric space, equipped with the metric obtained by restricting d.

Suppose we have two subsets A and B of X, which are both complete. Show that $A \cup B$ is complete.

Problem 2 (weight 20%)

Let X be a set, equipped with two metrics denoted d_1 and d_2. We define a function $\rho : X \times X \to \mathbb{R}_+$ by, for $(x, y) \in X \times X$:

$$\rho(x, y) = \max\{d_1(x, y), d_2(x, y)\}. \tag{1}$$

a (weight 10%)

Check that for any $(a_1, b_1, a_2, b_2) \in \mathbb{R}^4$:

$$\max\{a_1 + b_1, a_2 + b_2\} \le \max\{a_1, a_2\} + \max\{b_1, b_2\}. \tag{2}$$

Show that ρ is a metric on X.

(Continued on page 2.)

b (weight 10%)

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X, and let $a \in X$. Show that (u_n) converges to a with respect to ρ if and only if (u_n) converges to a with respect to both d_1 and d_2.

Problem 3 (weight 20%)

Let $X = \mathbb{R}^{n\times n}$ be the vector space of $n \times n$ real matrices, for some $n \in \mathbb{N}$ (with $n \geq 2$). We equip $\mathbb{R}^n$ with the Euclidean norm, denoted $|\cdot|$. We equip X with the corresponding operator norm, so that for any $M \in X$:

$$\|M\| = \sup_{u\in\mathbb{R}^n\setminus\{0\}} \frac{|Mu|}{|u|}. \tag{3}$$

We choose a (non-zero) matrix $J \in X$. We define a map $F : X \to X$ by, for $M \in X$:

$$F(M) = MJM. \tag{4}$$

a (weight 10%)

Show that F is differentiable on X and that for any $M \in X$, the differential of F at M, denoted by $\mathrm{D}F(M)$, is given by, for all $N \in X$:

$$\mathrm{D}F(M)N = MJN + NJM. \tag{5}$$

b (weight 10%)

Show that for any bounded subset A of X, F restricted to A is Lipschitz.

Problem 4 (weight 40%)

We let X be the space of continuous realvalued functions on $[0, 1]$:

$$X = \mathrm{C}([0, 1], \mathbb{R}). \tag{6}$$

We equip X with the supremum norm:

$$\|u\| = \sup\{|u(x)| \ : \ x \in [0, 1]\}. \tag{7}$$

Let $k : [0, 1] \times [0, 1] \to \mathbb{R}$ be a continuous function. We define an operator L on X as follows. Given $u \in X$ we let $Lu = v$ be the function $v : [0, 1] \to \mathbb{R}$ defined by, for $x \in [0, 1]$:

$$v(x) = \int_0^1 k(x, y)u(y)\mathrm{d}y. \tag{8}$$

a (weight 10%)

Justify that k is uniformly continuous. Use this to prove that for any $u \in X$, we have that Lu is continuous (that is $Lu \in X$).

(Continued on page 3.)

b (weight 10%)

We define:

$$C = \sup \left\{ \int_0^1 |k(x, y)| \mathrm{d}y \ : \ x \in [0, 1] \right\}. \tag{9}$$

Show that $C < +\infty$.

c (weight 10%)

Recall that for any bounded linear operator $T : X \to X$, its operator norm is defined by:

$$\|T\| = \sup_{u \in X \setminus \{0\}} \frac{\|Tu\|}{\|u\|}. \tag{10}$$

In what follows we use without proof that L is a linear map from X to X. Show that L is bounded, and that its operator norm is bounded by the previously introduced constant C, that is $\|L\| \leq C$.

d (weight 10%)

We suppose that k is defined by $k(x, y) = \sin(\pi x) \sin(\pi y)$. Let $I : X \to X$ be the identity operator. Show that the operator $I + L : X \to X$ is bijective and has a bounded inverse.

THE END