UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT2400 — Real Analysis

Day of examination: Friday 2. June 2023

Examination hours: 15:00 – 19:00

This problem set consists of 6 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

For each question you may use results from the preceding questions even if you haven't answered them.

Problem 1 (weight 20%)

Let (X, d) be a metric space.

a (weight 10%)

Suppose that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Suppose furthermore that for a certain $a \in X$, (u_n) has a subsequence that converges to a. Show that (u_n) converges to a.

Let $(u_{\phi(n)})$ denote the subsequence.
Let $\epsilon > 0$. Choose $m_1 \in \mathbb{N}$ such that for $n \geq m_1$ we have:

$$d(u_{\phi(n)}, a) < \epsilon/2. \tag{1}$$

Choose also $m_2 \in \mathbb{N}$ such that for $p, q \geq m_2$ we have:

$$d(u_p, u_q) < \epsilon/2. \tag{2}$$

Let $m = \max\{m_1, m_2\}$. For $n \geq m$ we have:

$$d(u_n, a) \leq d(u_n, u_{\phi(m)}) + d(u_{\phi(m)}, a), \tag{3}$$

$$< \epsilon/2 + \epsilon/2 = \epsilon. \tag{4}$$

We used that $\phi(m) \geq m$.

b (weight 10%)

In this question any subset of X is considered as a metric space, equipped with the metric obtained by restricting d.

(Continued on page 2.)

Suppose we have two subsets A and B of X, which are both complete. Show that $A \cup B$ is complete.

> Let (u_n) be a Cauchy sequence in $A \cup B$. Either A or B contains infinitely many terms. Suppose A has this property. Let (v_n) be the corresponding subsequence of (u_n). Then (v_n) is a Cauchy sequence [...]. Using that A is complete, let $a \in A$ be its limit. Then we are in the situation of the previous question. So (u_n) converges to a limit in $A \cup B$.

Problem 2 (weight 20%)

Let X be a set, equipped with two metrics denoted d_1 and d_2. We define a function $\rho : X \times X \to \mathbb{R}_+$ by, for $(x, y) \in X \times X$:

$$\rho(x, y) = \max\{d_1(x, y), d_2(x, y)\}. \tag{5}$$

a (weight 10%)

Check that for any $(a_1, b_1, a_2, b_2) \in \mathbb{R}^4$:

$$\max\{a_1 + b_1, a_2 + b_2\} \leq \max\{a_1, a_2\} + \max\{b_1, b_2\}. \tag{6}$$

Show that ρ is a metric on X.

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[3] We have:

$$a_1 \leq \max\{a_1, a_2\}, \tag{7}$$
$$b_1 \leq \max\{b_1, b_2\}. \tag{8}$$

Hence:

$$a_1 + b_1 \leq \max\{a_1, a_2\} + \max\{b_1, b_2\}. \tag{9}$$

Similarly:

$$a_2 + b_2 \leq \max\{a_1, a_2\} + \max\{b_1, b_2\}. \tag{10}$$

Combining we get the sought inequality.
[2] We have:

$$\rho(x, y) = 0 \iff d_1(x, y) = 0 \wedge d_2(x, y) = 0, \tag{11}$$
$$\iff x = y. \tag{12}$$

[2] We have:

$$\rho(x, y) = \max\{d_1(x, y), d_2(x, y)\}, \tag{13}$$
$$= \max\{d_1(y, x), d_2(y, x)\}, \tag{14}$$
$$= \rho(y, x). \tag{15}$$

[3] We have (using the inaugural question to get the second line):

$$\rho(x, z) \leq \max\{d_1(x, y) + d_1(y, z), d_2(x, y) + d_2(y, z)\}, \tag{16}$$
$$\leq \max\{d_1(x, y), d_2(x, y)\} + \max\{d_1(y, z), d_2(y, z)\}, \tag{17}$$
$$\leq \rho(x, y) + \rho(y, z). \tag{18}$$

b (weight 10%)

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X, and let $a \in X$. Show that (u_n) converges to a with respect to ρ if and only if (u_n) converges to a with respect to both d_1 and d_2.

[5] Suppose that (u_n) converges to a for ρ. Then $d_1(u_n, a)$ converges to 0 by the squeezing lemma, so (u_n) converges to a for d_1. The case of d_2 is similar.
[5] Suppose that (u_n) converges to a for d_1 and d_2. Let $\epsilon > 0$. For $i \in \{1, 2\}$ choose m_i such that for all all $n \geq m_i$ we have $d_i(u_n, a) < \epsilon$. Then for $n \geq \max\{m_1, m_2\}$ we have $\rho(u_n, a) < \epsilon$.

Problem 3 (weight 20%)

Let $X = \mathbb{R}^{n\times n}$ be the vector space of $n \times n$ real matrices, for some $n \in \mathbb{N}$ (with $n \geq 2$). We equip $\mathbb{R}^n$ with the Euclidean norm, denoted $|\cdot|$. We equip

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X with the corresponding operator norm, so that for any $M \in X$:

$$\|M\| = \sup_{u \in \mathbb{R}^n \setminus \{0\}} \frac{|Mu|}{|u|}. \tag{19}$$

We choose a (non-zero) matrix $J \in X$. We define a map $F : X \to X$ by, for $M \in X$:

$$F(M) = MJM. \tag{20}$$

a (weight 10%)

Show that F is differentiable on X and that for any $M \in X$, the differential of F at M, denoted by $\mathrm{D}F(M)$, is given by, for all $N \in X$:

$$\mathrm{D}F(M)N = MJN + NJM. \tag{21}$$

Fix $M \in X$. For all $N \in X$ we have:

$$F(M+N) = (M+N)J(M+N), \tag{22}$$
$$= MJM + MJN + NJM + NJN. \tag{23}$$

We remark that $F(M) = MJM$. Also, $N \mapsto MJN + NJM$ is linear and continuous (finite dimensions).
Finally for $N \neq 0$:

$$\frac{\|NJN\|}{\|N\|} \leq \frac{\|N\|\|J\|\|N\|}{\|N\|} = \|N\|\|J\| \to 0, \tag{24}$$

as $N \to 0$.

b (weight 10%)

Show that for any bounded subset A of X, F restricted to A is Lipschitz.

Let A be a bounded subset of X. Find $R > 0$ such that $A \subseteq B(0, R)$.
For $M \in B(0, R)$ and $N \in X$, we have:

$$\|\mathrm{D}F(M)N\| = \|MJN + NJM\|, \tag{25}$$
$$\leq 2R\|J\|\|N\|. \tag{26}$$

Remark that $B(0, R)$ is convex. We get, via the mean value theorem, that F is Lipschitz with constant $2R\|J\|$ on $B(0, R)$, hence also on A.
Also possible: a direct computation.

Problem 4 (weight 40%)

We let X be the space of continuous realvalued functions on $[0, 1]$:

$$X = \mathrm{C}([0, 1], \mathbb{R}). \tag{27}$$

(Continued on page 5.)

We equip X with the supremum norm:

$$\|u\| = \sup\{|u(x)| \ : \ x \in [0,1]\}. \tag{28}$$

Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function. We define an operator L on X as follows. Given $u \in X$ we let $Lu = v$ be the function $v : [0,1] \to \mathbb{R}$ defined by, for $x \in [0,1]$:

$$v(x) = \int_0^1 k(x,y)u(y)\mathrm{d}y. \tag{29}$$

a (weight 10%)

Justify that k is uniformly continuous. Use this to prove that for any $u \in X$, we have that Lu is continuous (that is $Lu \in X$).

Since $[0,1] \times [0,1]$ is compact and k is continuous, k is uniformly continuous.
Let $u \in X$. Let $\epsilon > 0$. Choose $\delta > 0$ such that for $|(x,y)-(x',y')| < \delta$ we have $|k(x,y) - k(x',y')| < \epsilon/\|u\|$.
We write, for $|x - x'| < \delta$:

$$|v(x) - v(x')| = |\int_0^1 (k(x,y) - k(x',y))u(y)\mathrm{d}y|, \tag{30}$$

$$\leq \int_0^1 |k(x,y) - k(x',y)||u(y)|\mathrm{d}y, \tag{31}$$

$$\leq \int_0^1 (\epsilon/\|u\|)\|u\|\mathrm{d}y = \epsilon. \tag{32}$$

This shows that v is uniformly continuous, hence continuous.

b (weight 10%)

We define:

$$C = \sup\left\{\int_0^1 |k(x,y)|\mathrm{d}y \ : \ x \in [0,1]\right\}. \tag{33}$$

Show that $C < +\infty$.

Since $[0,1] \times [0,1]$ is compact and k is continuous, we can find $M \in \mathbb{R}$ such that for all $x, y \in [0,1] \times [0,1]$ we have $|k(x,y)| \leq M$. Then we get, for any $x \in [0,1]$:

$$\int_0^1 |k(x,y)|\mathrm{d}y \leq M. \tag{34}$$

Hence $C \leq M < +\infty$.

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c (weight 10%)

Recall that for any bounded linear operator $T : X \to X$, its operator norm is defined by:

$$\|T\| = \sup_{u \in X \setminus \{0\}} \frac{\|Tu\|}{\|u\|}. \tag{35}$$

In what follows we use without proof that L is a linear map from X to X. Show that L is bounded, and that its operator norm is bounded by the previously introduced constant C, that is $\|L\| \leq C$.

We have:

$$\begin{aligned} \|Lu\| &\leq \sup_{x\in[0,1]} \left| \int_0^1 k(x,y)u(y)\mathrm{d}y \right|, & (36)\\ &\leq \sup_{x\in[0,1]} \int_0^1 |k(x,y)||u(y)|\mathrm{d}y, & (37)\\ &\leq \sup_{x\in[0,1]} \int_0^1 |k(x,y)|\mathrm{d}y \, \|u\|, & (38)\\ &\leq C\|u\|. & (39) \end{aligned}$$

d (weight 10%)

We suppose that k is defined by $k(x,y) = \sin(\pi x)\sin(\pi y)$. Let $I : X \to X$ be the identity operator. Show that the operator $I + L : X \to X$ is bijective and has a bounded inverse.

We compute:

$$\begin{aligned} \int_0^1 |k(x,y)|\mathrm{d}y &= \sin(\pi x) \int_0^1 \sin(\pi y)\mathrm{d}y, & (40)\\ &= \sin(\pi x) 2/\pi \leq 2/\pi. & (41) \end{aligned}$$

This shows that $\|L\| \leq 2/\pi < 1$. The theory of the Neumann series then gives the required conclusion.

THE END