UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Deferred exam in: MAT2400 — Real Analysis

Day of examination: Friday 18. August 2023

Examination hours: 09:00 – 13:00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

For each question you may use results from the preceding questions even if you haven't answered them.

Problem 1 (weight 40%)

For any function $f : [-\pi, \pi] \to \mathbb{C}$, with strong enough continuity properties, recall that the Fourier coefficients of f are given by, for $k \in \mathbb{Z}$:

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} \mathrm{d}t. \tag{1}$$

The Fourier series of f is the sequence of functions $f_n : [-\pi, \pi] \to \mathbb{C}$ given by for $n \in \mathbb{N}$ and $t \in [-\pi, \pi]$:

$$f_n(t) = \sum_{k=-n}^{n} a_k e^{ikt}. \tag{2}$$

In this problem we let f be defined by, for $t \in [-\pi, \pi]$:

$$f(t) = |t|. \tag{3}$$

a (weight 10%)

Compute the Fourier coefficients of f.

b (weight 10%)

Show that the Fourier series of f converges uniformly to some function.

c (weight 10%)

Show that the Fourier series of f converges uniformly to f.

(Continued on page 2.)

d (weight 10%)

Deduce that:

$$\sum_{l=0}^{\infty} \frac{1}{(2l+1)^2} = \frac{\pi^2}{8}. \tag{4}$$

Problem 2 (weight 30%)

Let X be a set equipped with a metric d. Let $\alpha \in \mathbb{R}$ be such that $0 < \alpha < 1$. Define a function $\rho : X \times X \to \mathbb{R}_+$ by, for all $x, y \in X$:

$$\rho(x, y) = (d(x, y))^\alpha. \tag{5}$$

a (weight 10%)

Check that for any $a, b \in \mathbb{R}$ such that $a \geq 0$, $b \geq 0$ and $a + b > 0$, we have:

$$1 \leq \left(\frac{a}{a+b}\right)^\alpha + \left(\frac{b}{a+b}\right)^\alpha. \tag{6}$$

Show that ρ is a metric on X.

b (weight 10%)

Let $\mathrm{id}_X : X \to X$ be the identity map. Show that id_X is continuous from (X, d) to (X, ρ) and from (X, ρ) to (X, d).

c (weight 10 %)

In this question we suppose that $X = [0, 1]$. We suppose furthermore that d is defined by, for $(x, y) \in X \times X$, we have $d(x, y) = |x - y|$.

Show that id_X is Lipschitz from (X, ρ) to (X, d) but not from (X, d) to (X, ρ).

Problem 3 (weight 30%)

Let $(a, b) \in \mathbb{R}^2$ with $(a, b) \neq (0, 0)$. We denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^2$:

$$|(x, y)| = (x^2 + y^2)^{1/2}. \tag{7}$$

We define U to be the open unit ball:

$$U = B(0, 1) = \{(x, y) \in \mathbb{R}^2 \ : \ |(x, y)| < 1\}. \tag{8}$$

Define $f : U \to \mathbb{R}$ by $f(0, 0) = 0$ and for $(x, y) \neq (0, 0)$:

$$f(x, y) = (ax + by)\frac{xy}{x^2 + y^2}. \tag{9}$$

a (weight 10%)

Justify that f is differentiable on $U \setminus \{(0, 0)\}$.

Check that the partial derivatives $\partial f(x, y)/\partial x$ and $\partial f(x, y)/\partial y$ are bounded on $U \setminus \{(0, 0)\}$ and deduce that the differential $\mathrm{D}f$ is bounded on $U \setminus \{(0, 0)\}$.

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b (weight 10%)

Show that f is not differentiable at $(0, 0)$.

c (weight 10%)

Show that f is Lipschitz continuous on U.

THE END