UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Deferred exam in: MAT2400 — Real Analysis
Day of examination: Friday 18. August 2023
Examination hours: 09:00 – 13:00
This problem set consists of 6 pages.
Appendices: None
Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

For each question you may use results from the preceding questions even if you haven't answered them.

Problem 1 (weight 40%)

For any function $f : [-\pi, \pi] \to \mathbb{C}$, with strong enough continuity properties, recall that the Fourier coefficients of f are given by, for $k \in \mathbb{Z}$:

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} \mathrm{d}t. \tag{1}$$

The Fourier series of f is the sequence of functions $f_n : [-\pi, \pi] \to \mathbb{C}$ given by for $n \in \mathbb{N}$ and $t \in [-\pi, \pi]$:

$$f_n(t) = \sum_{k=-n}^{n} a_k e^{ikt}. \tag{2}$$

In this problem we let f be defined by, for $t \in [-\pi, \pi]$:

$$f(t) = |t|. \tag{3}$$

a (weight 10%)

Compute the Fourier coefficients of f.

First of all $a_0 = \pi/2$.
We compute for $k \neq 0$:

$$\int_0^{\pi} t e^{-ikt} \mathrm{d}t = -\pi \frac{(-1)^k}{ik} + \frac{(-1)^k - 1}{k^2}. \tag{4}$$

From there we get:

$$a_k = \frac{(-1)^k - 1}{\pi k^2}. \tag{5}$$

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b (weight 10%)

Show that the Fourier series of f converges uniformly to some function.

> The series $\sum_k 1/k^2$ converges. Hence the Fourier series of f converges uniformly according to the Weierstrass M-test.

c (weight 10%)

Show that the Fourier series of f converges uniformly to f.

> Let's say (f_n) converges uniformly to the function g. Then (f_n) converges in mean quare norm to g also. But on the other hand we know that (f_n) converges in mean square norm to f. By uniqueness of limits $g = f$. So (f_n) converges uniformly to f.

d (weight 10%)

Deduce that:

$$\sum_{l=0}^{\infty} \frac{1}{(2l+1)^2} = \frac{\pi^2}{8}. \tag{6}$$

> We evaluate the Fourier series in 0. We get:
>
> $$0 = \frac{\pi}{2} + 2\sum_{k=1}^{\infty} \frac{(-1)^k - 1}{\pi k^2}, \tag{7}$$
> $$= \frac{\pi}{2} + 2\sum_{l=0}^{\infty} \frac{-2}{\pi(2l+1)^2}. \tag{8}$$
>
> This gives the identity.

Problem 2 (weight 30%)

Let X be a set equipped with a metric d. Let $\alpha \in \mathbb{R}$ be such that $0 < \alpha < 1$. Define a function $\rho : X \times X \to \mathbb{R}_+$ by, for all $x, y \in X$:

$$\rho(x, y) = (d(x, y))^\alpha. \tag{9}$$

a (weight 10%)

Check that for any $a, b \in \mathbb{R}$ such that $a \geq 0$, $b \geq 0$ and $a + b > 0$, we have:

$$1 \leq \left(\frac{a}{a+b}\right)^\alpha + \left(\frac{b}{a+b}\right)^\alpha. \tag{10}$$

Show that ρ is a metric on X.

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[3] We have:

$$0 \leq \frac{a}{a+b} \leq 1, \tag{11}$$

hence:

$$\frac{a}{a+b} \leq \left(\frac{a}{a+b}\right)^{\alpha}. \tag{12}$$

Similarly we have:

$$\frac{b}{a+b} \leq \left(\frac{b}{a+b}\right)^{\alpha}. \tag{13}$$

Adding we get:

$$1 \leq \left(\frac{a}{a+b}\right)^{\alpha} + \left(\frac{b}{a+b}\right)^{\alpha}. \tag{14}$$

[2] It is clear that $\rho(x, y) \geq 0$. Moreover $\rho(x, y) = 0$ iff $d(x, y) = 0$, which is equivalent to $x = y$.
[2] Symmetry of ρ is clear from the symmetry of d.
[3] Triangle inequality:

$$\begin{align}
\rho(x, z) &= (d(x, z))^{\alpha}, \tag{15}\\
&\leq (d(x, y) + d(y, z))^{\alpha}, \tag{16}\\
&\leq d(x, y)^{\alpha} + d(y, z)^{\alpha}, \tag{17}\\
&= \rho(x, y) + \rho(y, z). \tag{18}
\end{align}$$

We used first that $t \mapsto t^{\alpha}$ is increasing, then the first question.

b (weight 10%)

Let $\mathrm{id}_X : X \to X$ be the identity map. Show that id_X is continuous from (X, d) to (X, ρ) and from (X, ρ) to (X, d).

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X and $a \in X$.
If $x_n \to a$ wrt d, then $d(x_n, a) \to 0$, then $\rho(x_n, a) \to 0$ so $x_n \to a$ wrt ρ. This shows that id_X is continuous from (X, d) to (X, ρ).
The other way is similar.

c (weight 10 %)

In this question we suppose that $X = [0, 1]$. We suppose furthermore that d is defined by, for $(x, y) \in X \times X$, we have $d(x, y) = |x - y|$.

Show that id_X is Lipschitz from (X, ρ) to (X, d) but not from (X, d) to (X, ρ).

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[5] For $x, y \in [0,1]$ we have $|x-y| \leq 1$, hence:

$$d(x,y) = |x-y| \leq |x-y|^\alpha = \rho(x,y). \tag{19}$$

This shows that id_X is Lipschitz from (X,ρ) to (X,d).
[5] Suppose that id_X is Lipschitz from (X,d) to (X,ρ). Choose $C \geq 0$ such that:

$$\forall x, y \in X \quad \rho(x,y) \leq C d(x,y). \tag{20}$$

We get especially:

$$|x|^\alpha \leq C|x|. \tag{21}$$

This contradicts the fact that:

$$\lim_{x \to 0^+} x^{\alpha-1} = +\infty. \tag{22}$$

Problem 3 (weight 30%)

Let $(a,b) \in \mathbb{R}^2$ with $(a,b) \neq (0,0)$. We denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^2$:

$$|(x,y)| = (x^2+y^2)^{1/2}. \tag{23}$$

We define U to be the open unit ball:

$$U = B(0,1) = \{(x,y) \in \mathbb{R}^2 \;:\; |(x,y)| < 1\}. \tag{24}$$

Define $f : U \to \mathbb{R}$ by $f(0,0) = 0$ and for $(x,y) \neq (0,0)$:

$$f(x,y) = (ax+by)\frac{xy}{x^2+y^2}. \tag{25}$$

a (weight 10%)

Justify that f is differentiable on $U \setminus \{(0,0)\}$.

Check that the partial derivatives $\partial f(x,y)/\partial x$ and $\partial f(x,y)/\partial y$ are bounded on $U \setminus \{(0,0)\}$ and deduce that the differential $\mathrm{D}f$ is bounded on $U \setminus \{(0,0)\}$.

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The function f has continuous partial derivatives on $U \setminus \{(0,0)\}$ by the standard theorems, hence is differentiable on $U \setminus \{(0,0)\}$.
We get:

$$\frac{\partial f(x,y)}{\partial x} = a\frac{xy}{x^2+y^2} + (ax+by)\frac{(y^2-x^2)y}{(x^2+y^2)^2}. \tag{26}$$

We remark that, in terms of the Euclidean norm, $x^2 + y^2 = |(x,y)|^2$ and that $|(x,y)|$ bounds both $|x|$ and $|y|$.
This shows that $\frac{\partial f(x,y)}{\partial x}$ is bounded on $U \setminus \{(0,0)\}$. Similarly for $\frac{\partial f(x,y)}{\partial y}$.
We have:

$$\|\mathrm{D}f(x.y)\| \leq |(\frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y})|, \tag{27}$$

by the Cauchy-Schwarz inequality on $\mathbb{R}^2$. Hence $\mathrm{D}f$ is bounded on $U \setminus \{(0,0)\}$.
It follows that $\mathrm{D}f$ is also bounded on $U \setminus \{(0,0)\}$.

b (weight 10%)

Show that f is not differentiable at $(0,0)$.

We compute:

$$\frac{f(tx,ty) - f(0,0)}{t} = (ax+by)\frac{xy}{x^2+y^2}. \tag{28}$$

We let $t \to 0$ to obtain the directional derivatives at 0. These do not depend linearly on (x,y), hence the function is not differentiable at 0.

c (weight 10%)

Show that f is Lipschitz continuous on U.

We remark that f is continuous at $(0,0)$, hence everywhere on U.
Let C be a bound on $\|\mathrm{D}f\|$ on $U \setminus \{(0,0)\}$ with respect to the Euclidean norm.
Let (x,y) and (x',y') be two points in U.
Suppose that $(0,0)$ does not belong to the open segment from (x,y) to (x',y'). Then the mean value inequality applies and gives:

$$|f(x',y') - f(x,y)| \leq C|(x'-x, y'-y)|. \tag{29}$$

Suppose on the other hand that $(0,0)$ belongs to this open segment. We apply the preceding inequality to the two line segments joining $(0,0)$ to the endpoints. We get:

$$\begin{aligned} |f(x',y') - f(x,y)| &\leq |f(x',y')| + |f(x,y)|, &(30)\\ &\leq C|(x',y')| + C|(x,y)|, &(31)\\ &\leq C|(x'-x, y'-y)|. &(32) \end{aligned}$$

(Continued on page 6.)

THE END