

MAT2400, Spring 2024

Mandatory assignment 1 of 2

Submission deadline

Thursday 22 February 2024, 23:59 in Canvas (canvas.uio.no).

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LATEX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course, and assignment number.

It is expected that you give a clear presentation with all necessary explanations. You only have one attempt at each assignment, and you need to have both assignments approved in order to take the exam. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Problem 1

(a)

We have established through an exercise that if a sequence $\{x_n\}$ converges to $a \in \mathbb{R}$, then all its subsequences also converge to a . Prove that if all subsequences of a sequence $\{x_n\}$ converge to the *same* limit a , then the sequence $\{x_n\}$ itself converges to a .

Answer: This is trivial if you accept that a sequence can be considered a subsequence of itself, analogous to the concept that any set is its own subset. Alternatively, assume for a contradiction that $\{x_n\}$ does not converge to a . This implies that there exists an $\varepsilon > 0$ such that, for any integer N (no matter how large!), there exists an $n \geq N$ such that $|x_n - a| \geq \varepsilon$. Given this, we can create a subsequence $\{x_{n_k}\}$ from the terms x_n that do not satisfy the condition $|x_n - a| < \varepsilon$: indeed, for each index k pick an element x_{n_k} from the original sequence $\{x_n\}$ such that $|x_{n_k} - a| \geq \varepsilon$. This new subsequence will not converge to a , contradicting our assumption that all subsequences converge to a . Thus, the original sequence $\{x_n\}$ must converge to a .

(b)

Let F be a nonempty set that is both closed and bounded in $\mathbb{R}$, using the standard metric $d(x, y) = |y - x|$. Use the Bolzano-Weierstrass theorem to demonstrate that the supremum of F belongs to F , $\sup F \in F$. Additionally, provide an example of a set F that is bounded but not closed, where both $\inf F \notin F$ and $\sup F \notin F$.

Answer: Set $s = \sup F$. By definition of supremum, for each $n \in \mathbb{N}$, there exists $x_n \in F$ such that

$$s - \frac{1}{n} < x_n \leq s,$$

so that $x_n \rightarrow s$ as $n \rightarrow \infty$. The sequence $\{x_n\}$ constructed this way is a bounded sequence in F and by the Bolzano-Weierstrass theorem, it has a convergent subsequence $\{x_{n_k}\}$ that converges to a limit $l \in \mathbb{R}$. Since F is closed, it contains all its limit points and hence $l \in F$. The limit of this subsequence is also the limit of the original sequence, and thus

$$s = \lim_{k \rightarrow \infty} x_{n_k} = l \in F.$$

Example: Consider the set $F = (0, 1)$, which is a bounded open interval in $\mathbb{R}$. We have $\sup F = 1 \notin F$ and $\inf F = 0 \notin F$.

(c)

Let (X, d) be a metric space, and consider a subset A of X , equipped with the subset metric d_A . You may want to use the notations $B_X(x, r)$ and $B_A(x, r)$ for the open balls in (X, d) and (A, d_A) , respectively.

(i) If a subset $O \subset A$ is open in the metric space (X, d) , then show that O is also open in the subset (A, d_A) .

(ii) Provide an example to illustrate that a subset O of A , while being open in (A, d_A) , may not necessarily be open in the larger space (X, d) .

Answer: (i) Given that $O \subseteq A$ is open in (X, d) , for any point $x \in O$, there exists an $\varepsilon > 0$ such that $B_X(x, \varepsilon) \subset O$. For the subset metric d_A , the open ball is defined as

$$B_A(x, \varepsilon) = \{a \in A : d(x, a) < \varepsilon\} = B_X(x, \varepsilon) \cap A;$$

clearly, $B_A(x, \varepsilon) \subset O$, so O is open in (A, d_A) .

(ii) Consider $X = \mathbb{R}$, the real numbers with the standard metric, and let $A = [0, 1) \subset X$. Define $O = [0, \frac{1}{2})$. The set O is open in the subspace topology of A since for any $x \in O$, there exists an $\varepsilon > 0$ such that $B_A(x, \varepsilon) = (x - \varepsilon, x + \varepsilon) \cap A \subset O$. In particular, $B_A(0, \varepsilon) = [0, \varepsilon) \subset O$ if $\varepsilon < 1/2$. However, O is not open in the full metric space (X, d) because it includes the point 0 which is the endpoint of A and does not have an open interval (ball) entirely contained within O .

(d)

Let (X, d) be a metric space, and consider a subset A of X , equipped with the subset metric d_A . Suppose A is an open set in (X, d) . Prove that if $O \subset A$ is open in (A, d_A) , then O is open in the larger space (X, d) .

Answer: If A is open in (X, d_X) , then for any $x \in A$, there exists $\varepsilon_A > 0$ such that $B_X(x, \varepsilon_A) \subset A$. If O is open in (A, d_A) , then for any $x \in O$, there exists an $\varepsilon_O > 0$ such that $B_A(x, \varepsilon_O) = B_X(x, \varepsilon_O) \cap A \subset O$. Set $\varepsilon := \min(\varepsilon_A, \varepsilon_O)$. Then $B_X(x, \varepsilon) \subset O$ and thus x is an interior point of O in X . By the arbitrariness of $x \in O$, this implies that O is open in X .

Problem 2

(a)

We say that two metrics d_1 and d_2 on X are *equivalent* if they induce the same topology, that is, a sequence $\{x_n\}$ converges to x with respect to d_1 if and only if it converges to x with respect to d_2 .

Consider a metric space (X, d) , and introduce another metric (on X) defined by

$$d_m(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \quad x, y \in X.$$

Show that d and d_m are equivalent.

Hint: To establish the implication “convergence with respect to d_m implies convergence with respect to d ”, use that if a positive number a satisfies $\frac{a}{1+a} < \frac{\varepsilon}{1+\varepsilon}$, then it follows that $a < \varepsilon$.

Answer: Assume $\{x_n\}$ converges to x with respect to d , then for every $\varepsilon > 0$, there exists N such that for all $n \geq N$, $d(x_n, x) < \varepsilon$. Then it follows that

$$d_m(x_n, x) = \frac{d(x_n, x)}{1 + d(x_n, x)} < \varepsilon;$$

hence, $\{x_n\}$ converges to x with respect to d_m .

Conversely, assume $\{x_n\}$ converges to x with respect to d_m , then for every $\varepsilon > 0$, there exists N such that for all $n \geq N$, $d_m(x_n, x) < \frac{\varepsilon}{1+\varepsilon}$:

$$d_m(x_n, x) = \frac{d(x_n, x)}{1 + d(x_n, x)} < \frac{\varepsilon}{1 + \varepsilon}.$$

This implies that

$$d(x_n, x) < \varepsilon.$$

Thus, $\{x_n\}$ converges to x with respect to d .

(b)

Suppose d_1 and d_2 are metrics on X . The equivalence of d_1 and d_2 can also be expressed by the existence of constants K and M satisfying

$$d_1(x, y) \leq K d_2(x, y), \quad d_2(x, y) \leq M d_1(x, y), \quad \forall x, y \in X.$$

Show that if d_1 and d_2 are equivalent, and (X, d_1) is complete, then (X, d_2) is also complete.

Answer: Since (X, d_1) is complete, every Cauchy sequence with respect to d_1 converges. Let $\{x_n\}$ be a Cauchy sequence in (X, d_2) . For every $\varepsilon > 0$, there exists an N such that for all $m, n \geq N$,

$$d_2(x_m, x_n) < \frac{\varepsilon}{K}.$$

From the equivalence of the metrics,

$$d_1(x_m, x_n) \leq K d_2(x_m, x_n) < \varepsilon, \quad \forall m, n \geq N.$$

Thus, $\{x_n\}$ is a Cauchy sequence in (X, d_1) . Since (X, d_1) is complete, there exists $x \in X$ to which $\{x_n\}$ converges with respect to d_1 . This implies the existence of an N such that for all $n \geq N$,

$$d_1(x_n, x) < \frac{\varepsilon}{M}.$$

Again using the equivalence of metrics,

$$d_2(x_n, x) \leq M d_1(x_n, x) < \varepsilon, \quad \forall n \geq N.$$

This establishes that $\{x_n\}$ converges to x with respect to d_2 , thus (X, d_2) is complete as well.

(c)

Prove that if two functions f and g , mapping from $\mathbb{R}$ to $\mathbb{R}$, are continuous at a specific point a , then the function defined by their product, $h(x) = f(x)g(x)$, is also continuous at the same point a .

Answer: To prove that h is continuous at a , we must show that for any arbitrary $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x satisfying $|x - a| < \delta$, the condition $|h(x) - h(a)| < \varepsilon$ holds.

Since f and g are continuous at a , for any given $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that:

1. $|f(x) - f(a)| < \varepsilon_1$ for all x where $|x - a| < \delta_1$.
2. $|g(x) - g(a)| < \varepsilon_2$ for all x where $|x - a| < \delta_2$.

Similarly, there exist $\tilde{\delta}_1 > 0$ such that:

$$|f(x) - f(a)| < 1 \quad \text{for all } x \text{ where } |x - a| < \tilde{\delta}_1.$$

By the inverse triangle inequality, this implies that

$$||f(x)| - |f(a)|| \leq |f(x) - f(a)| < 1,$$

and therefore

$$|f(x)| \leq |f(a)| + 1.$$

Let $\delta = \min\{\delta_1, \tilde{\delta}_1, \delta_2\}$. Then, for any x with $|x - a| < \delta$, all of the above inequalities are satisfied.

Now, consider the expression $|h(x) - h(a)|$:

$$\begin{aligned} |h(x) - h(a)| &= |f(x)g(x) - f(a)g(a)| \\ &= |f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)| \\ &\leq |f(x)||g(x) - g(a)| + |g(a)||f(x) - f(a)| \quad (\text{triangle inequality}). \end{aligned}$$

Since f is continuous at a , we have that $f(x)$ is bounded near a ; specifically, for $|x - a| < \delta$, we have that

$$|f(x)| < M, \quad M := \max\{|f(a)|, |g(a)|\} + 1 > 0.$$

Clearly, we also have

$$|g(a)| < M.$$

Applying these bounds gives us:

$$\begin{aligned} |h(x) - h(a)| &\leq |f(x)||g(x) - g(a)| + |g(a)||f(x) - f(a)| \\ &< M\varepsilon_2 + M\varepsilon_1. \end{aligned}$$

Now specifying

$$\varepsilon_1 = \frac{\varepsilon}{2M}, \quad \varepsilon_2 = \frac{\varepsilon}{2M},$$

gives

$$|h(x) - h(a)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(d)

Let (X, d) be a metric space and let $f : X \rightarrow [0, \infty)$ be a continuous function. For each $\varepsilon > 0$, suppose there exists a compact subset $K_\varepsilon \subseteq X$ such that for all $x \notin K_\varepsilon$, $f(x) > \varepsilon$. Prove that f has a minimum point.

Hint: First, argue for the existence of the infimum of the forward image $f(X)$. Second, the hypothesis implies that for any $\varepsilon > 0$, the values of f outside the compact set K_ε are greater than ε . This means that to find the minimum value of f , it suffices to look within compact subsets of X , where f attains its minimum due to the continuity of f and the compactness of

the subsets (explain why).

Answer: Since $f(x) \geq 0 \forall x \in X$ (by assumption), the forward image

$$f(X) = \{f(x) \mid x \in X\} \subset \mathbb{R}$$

is bounded below by 0. The lower bound of $f(X)$ and the completeness of $\mathbb{R}$ imply that there exists an infimum: $\alpha = \inf f(X) \geq 0$.

According to the hypothesis, there exists a compact set $K_{\alpha+1/n} \neq \emptyset$ such that for all $x \notin K_{\alpha+1/n}$, $f(x) > \alpha + 1/n$, for each $n = 1, 2, \dots$. However, inside $K_{\alpha+1/n}$, the function f must attain a minimum value $m_n (\geq 0)$ due to the compactness of $K_{\alpha+1/n}$ and the continuity of f . Indeed, $f(K_{\alpha+1/n}) \subset [0, \infty)$ is compact in $\mathbb{R}$. As a result, $f(K_{\alpha+1/n})$ attains its infimum at some point $f(x_n)$, where $x_n \in K_{\alpha+1/n}$: $f(x_n) = \inf f(K_{\alpha+1/n}) = m_n$. Since $K_{\alpha+1/n}$ contains all points x where $f(x) \leq \alpha + 1/n$, it follows that $f(x_n) = m_n \leq \alpha + 1/n$. At the same time, we know that $\alpha \leq f(x_n)$. Consequently, $\alpha \leq f(x_n) \leq \alpha + 1/n$, so that $f(x_n) \rightarrow \alpha$ as $n \rightarrow \infty$. Observe that $\{x_n\} \subset K_{\alpha+1}$ (as $\alpha + 1/n \leq \alpha + 1$ for all n). Since $K_{\alpha+1}$ is compact, there exists a subsequence $\{x_{n_k}\}$ and $x_0 \in K_{\alpha+1}$ such that $x_{n_k} \rightarrow x_0$. By the continuity of f , $f(x_{n_k}) \rightarrow f(x_0) = \alpha$. In other words, f attains its minimum at x_0 .

Shorter argument: Let $\alpha = \inf f(X) \geq 0$ (which exists by the argument above). Then there exists $x_n \in X$ such that $\alpha \leq f(x_n) < \alpha + 1/n$, for each $n = 1, 2, \dots$. This implies that $\{x_n\} \subset K_{\alpha+1}$ (as $\alpha + 1/n \leq \alpha + 1$). Since $K_{\alpha+1}$ compact, there exists a subsequence $\{x_{n_k}\}$ and $x_0 \in K_{\alpha+1}$ such that $x_{n_k} \rightarrow x_0$. By the continuity of f , $f(x_{n_k}) \rightarrow f(x_0) = \alpha$. In other words, f attains its minimum at x_0 .