February 26, 2026

MAT2400 – Real Analysis

Mandatory assignment 1 of 2

Submission deadline

Thursday 12 March 2026 at 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between writing in English or Norwegian. Use the Norwegian–English mathematical dictionary to look up terms.

You can choose between scanning handwritten notes or writing the solution directly on a computer with LaTeX. The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible; please use either the "Color" or "Grayscale" settings. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures.

Note that you have **one** attempt to pass the assignment. This means that there are no second attempts. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

Read the department's guidelines for mandatory assignments for more information.

Introduction

The *calculus of variations* is the mathematical theory of finding minima (or maxima) of functions. Such problems arise in a very large number of applications: What is the shortest path from A to B? What is the optimal way of investing your savings in the stock market? What are the best weights in a deep neural network, i.e., the weights that minimize a given loss function? How large should fishing quotas be, and where should they be distributed, in order to maximize both the profit and sustainability of fish stocks?

The extreme value theorem tells us that if a function $f\colon E \to \mathbb{R}$ is continuous and E is compact, then f attains a minimum in E. (We will not get into the details of how to find this minimum here.) However, for many applications, f will not be continuous, and E will not be compact. What will often be the case, however, is that E is *precompact*, and f is *lower semicontinuous*. This is enough to prove that f attains a minimum somewhere in $\overline{E}$ (the closure of E).

The goal of this mandatory assignment is to carry out parts of this program for the following problem:

> find the curve that starts at a given point $x_0 \in \mathbb{R}^m$, ends at another point $x_1 \in \mathbb{R}^m$, and which has the smallest cost.

We will measure the "cost" of going through a point $x \in \mathbb{R}^m$ by a function $k = k(x) \in [0, \infty)$. (The number $k(x)$ could for instance indicate how strenuous it is to walk over x, or how much time it takes to walk one meter near x.) The *cost* of a curve parameterized by a function $x\colon [0,1] \to \mathbb{R}^m$ is then

$$f(x) \coloneqq \int_0^1 k(x(t)) \left\| \frac{dx}{dt}(t) \right\| dt,$$

the curve integral of the function k along the curve. Your goal will be to prove the existence of a minimiser of f, i.e., an "optimal" curve.

Problems

Note: *In order to get this mandatory assignment approved, you need to attempt to solve at least 5 of the following 10 sub-problems. By an "attempt" we mean a written answer that shows that you have made an honest attempt at solving the sub-problem.*

Problem 1. Let (X, d) be a metric space and $K \subseteq X$ a subset. A function $f\colon K \to \mathbb{R}$ is *lower semicontinuous* if for all $x \in K$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$y \in K,\ d(x,y) < \delta \qquad \Longrightarrow \qquad f(x) < f(y) + \varepsilon.$$

The goal of this problem is to show that if $f\colon K \to \mathbb{R}$ is lower semicontinuous and K is compact, then f attains a minimum, i.e. there is some $\bar{x} \in K$ such that

$$f(\bar{x}) \leqslant f(x) \qquad \forall\ x \in K.$$

You may proceed as follows:

(a) Let $m \coloneqq \inf\{f(x) : x \in K\}$. Argue that there exists a *minimizing sequence* for f, i.e. a sequence $(x_n)_{n\in\mathbb{N}}$ in K such that $f(x_n) \to m$ as $n \to \infty$.

Assume from now on that K is compact.

(b) Show that there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and some $\bar{x} \in K$ such that $x_{n_k} \to \bar{x}$ as $k \to \infty$.

(c) Show that $m \leqslant f(\bar{x}) \leqslant m + \varepsilon$ for any $\varepsilon > 0$, and conclude that $\bar{x}$ is a minimum of f.

Solution:

(a) If $m = -\infty$, then we let $(x_n)_n$ be any sequence such that e.g. $f(x_n) < -n$ for all $n \in \mathbb{N}$. (Such a point x_n must exist, for otherwise $-n$ would be a greater lower bound.) If $m \in \mathbb{R}$ then, by virtue of being the greatest lower bound of $\{f(x) : x \in K\}$, there must for every $n \in \mathbb{N}$ be some $\alpha \in \{f(x) : x \in K\}$ such that $m \geqslant \alpha - 1/n$ (otherwise, $m + 1/n$ would be a greater lower bound). Let $x_n \in K$ be a point where $\alpha = f(x_n)$. Then $m \leqslant f(x_n) \leqslant m + 1/n$ for all n, so $f(x_n) \to m$.

(b) This follows from K being compact.

(c) Let $\varepsilon > 0$, and let $\delta > 0$ be as in the definition of lower semicontinuity. Let $(x_{n_k})_{k\in\mathbb{N}}$ be the subsequence converging to $\bar{x}$. Let $N \in \mathbb{N}$ be such that if $k \geqslant N$ then $d(x_{n_k}, \bar{x}) < \delta$. Then

$$m \leqslant f(\bar{x}) < f(x_{n_k}) + \varepsilon \qquad \forall\, k \geqslant N. \tag{$*$}$$

Since $f(x_n) \to m$, we also have $f(x_{n_k}) \to m$. Passing $k \to \infty$ in $(*)$, we get

$$m \leqslant f(\bar{x}) \leqslant m + \varepsilon.$$

Since ε was arbitrary, this implies $f(\bar{x}) = m$, whence $\bar{x}$ is a minimum for f. In particular, since $f(\bar{x})$ is a real number, the infimum m is finite.

Problem 2. Let (X, d) be a metric space and $E \subseteq X$ any subset. Let $f\colon E \to \mathbb{R}$ be any bounded function (i.e., there is some $M \in \mathbb{R}$ such that $|f(x)| \leqslant M$ for all $x \in E$). Define $g\colon \overline{E} \to \mathbb{R}$ by

$$g(x) \coloneqq \liminf_{\substack{y\to x \\ y\in E}} f(y) \qquad \forall\, x \in \overline{E},$$

that is,

$$g(x) \coloneqq \lim_{n\to\infty} f_n(x), \qquad \text{where } f_n(x) \coloneqq \inf\big\{f(y) : y \in E \cap B(x; 1/n)\big\}.$$

(Here, $\overline{E} \coloneqq E \cup \partial E$ is the closure of E.) The function g is called the *lower semicontinuous envelope of f*.

(a) Show that g is well-defined, i.e., that the above limit exists, and that

$$g(x) \leqslant f(x) \tag{1}$$

for all $x \in E$.

(b) Show that g is lower semicontinuous.

(c) Show that if f is lower semicontinuous on E, then $g(x) = f(x)$ for all $x \in E$.

Solution:

(a) For each $x \in \overline{E}$ and $n \in \mathbb{N}$, the set $D_n \coloneqq \{f(y) : y \in E \cap B(x; 1/n)\}$ is non-empty and bounded (since f is bounded), so it has an infimum $f_n(x)$. The sequence $(f_n(x))_{n\in\mathbb{N}}$ is increasing, since if $n \geqslant n'$ then

$$D_n \subseteq D_{n'},$$

so that $f_n(x) \coloneqq \inf D_n \geqslant \inf D_{n'} =: f_{n'}(x)$. We also have $f_n(x) \leqslant M$ for all n, so $(f_n(x))_{n\in\mathbb{N}}$ is a bounded, increasing sequence, whence it has a limit (by the completeness principle, Theorem 2.2.2 in *Spaces*).

For $x \in E$ we have $f_n(x) \leqslant f(x)$, by the definition of $f_n(x)$, so

$$g(x) = \lim_{n\to\infty} f_n(x) \leqslant f(x) \qquad \forall\ x \in E,$$

which proves (1).

(b) Let $x \in \overline{E}$, and let $\varepsilon > 0$. Since $f_n(x) \to g(x)$, there is some $N \in \mathbb{N}$ such that $|f_n(x) - g(x)| < \varepsilon$ when $n \geqslant N$. Letting $\delta \coloneqq 1/n$, we also note that for every $y \in E \cap B(x; \delta)$, we have

$$f_n(x) \leqslant f_k(y)$$

if $k \in \mathbb{N}$ is large enough that $B(y; 1/k) \subseteq B(x; 1/n)$ (take e.g. k such that $1/k \leqslant 1/n - d(x, y)$). Letting $k \to \infty$ then yields

$$f_n(x) \leqslant g(y) \qquad \forall\ y \in E \cap B(x; \delta).$$

Hence,

$$g(x) \leqslant f_n(x) + |g(x) - f_n(x)| < f_n(x) + \varepsilon \leqslant g(y) + \varepsilon$$

for all $y \in E \cap B(x; \delta)$. This proves that g is lower semicontinuous.

(c) Let $x \in E$ and let $\varepsilon > 0$. Since f is lower semicontinuous at x, there is some $\delta > 0$ such that

$$f(x) - f(y) < \varepsilon$$

for all $y \in E \cap B(x;\delta)$. Let $n \geqslant 1/\delta$ (so that $1/n \leqslant \delta$, and therefore $B(x;1/n) \subseteq B(x;\delta)$), and take the infimum of the above inequality over all $y \in E \cap B(x;1/n)$ to get

$$f(x) - f_n(x) < \varepsilon.$$

Now let $n \to \infty$ to get

$$f(x) - g(x) \leqslant \varepsilon.$$

But we also know that $f(x) - g(x) \geqslant 0$ for all x. Since ε was arbitrary, we conclude that $f(x) = g(x)$.

Problem 3. Fix some $m \in \mathbb{N}$. We equip $\mathbb{R}^m$ with the metric given by the Euclidean norm, $d(x,y) = \|x - y\|$, where $\|z\| = \sqrt{|z_1|^2 + \cdots + |z_m|^2}$ (for $x, y, z \in \mathbb{R}^m$). Define

$$X \coloneqq \mathrm{Lip}([0,1], \mathbb{R}^m) = \big\{x\colon [0,1] \to \mathbb{R}^m : x \text{ is Lipschitz continuous}\big\},$$

$$Y \coloneqq C^1([0,1], \mathbb{R}^m) = \Big\{x\colon [0,1] \to \mathbb{R}^m : \text{both } x \text{ and } \tfrac{dx}{dt} \text{ exist and are continuous}\Big\}.$$

We equip X with the supremum metric,

$$\rho(x,y) \coloneqq \sup_{t\in[0,1]} |x(t) - y(t)| \qquad \text{for } x, y \in X.$$

Fix $x_0, x_1 \in \mathbb{R}^m$, let $c > 0$, and define

$$E_c \coloneqq \big\{x \in Y : x(0) = x_0,\ x(1) = x_1,\ \big\|\tfrac{dx}{dt}(t)\big\| \leqslant c\ \forall\ t \in [0,1]\big\}.$$

(a) Show that

(i) every $x \in E_c$ has Lipschitz constant at most c,

(ii) E_c is a subset of X,

(iii) E_c is non-empty when c is large enough.

(b) Show that E_c is a *precompact* subset of X, i.e., every sequence $(x_n)_{n\in\mathbb{N}}$ in E_c has a subsequence that converges to a point in X.

(c) For some continuous, bounded function $v\colon \mathbb{R}^m \to [0,\infty)$, define $f\colon E_c \to \mathbb{R}$ by

$$f(x) \coloneqq \int_0^1 v(x(t))\big\|\tfrac{dx}{dt}(t)\big\|\,dt \qquad \text{for } x \in E_c.$$

Show that f is bounded on E_c.

(d) It is a fact that f is not continuous on E_c, and E_c is not compact, so we cannot apply the Extreme value theorem to deduce the existence of a minimum of f.

It can be shown that f is lower semicontinuous (but you are not asked to show this here). Assuming that this is true, show that there is some $\bar{x} \in \overline{E}_c$ which "minimizes" f, in the sense that $\bar{x}$ is a minimum for the lower semicontinuous envelope g of f, and that if $\bar{x} \in E_c$, then $\bar{x}$ is a minimum of f.

Solution:

(a) Let $x \in E_c$. Let $s, t \in [0,1]$, and assume that, say, $s < t$. Then, by the fundamental theorem of calculus,

$$\|x(t) - x(s)\| = \left\| \int_s^t \frac{dx}{dt}(\tau)\, d\tau \right\| \leqslant \int_s^t \left\| \frac{dx}{dt}(\tau) \right\| d\tau \leqslant c(t-s).$$

This proves (i), and as a consequence, (ii).

If $c \geqslant \|x_0 - x_1\|$, let $x(t) \coloneqq x_0 + t(x_1 - x_0)$. Then

$$\left\| \frac{dx}{dt}(t) \right\| = \|x_1 - x_0\| \leqslant c \qquad \forall\, t \in [0,1],$$

so x lies in E_c. This proves (iii).

(b) All $x \in E_c$ have Lipschitz constant at most c, so E_c is equicontinuous. The set is also bounded, since for every $t \in [0,1]$,

$$\|x(t)\| = \left\| x_0 + \int_0^t \frac{dx}{dt}(\tau)\, d\tau \right\| \leqslant \|x_0\| + \int_0^t \left\| \frac{dx}{dt}(\tau) \right\| d\tau \leqslant \|x_0\| + tc \leqslant \|x_0\| + c.$$

Hence, $\rho(x, 0) \leqslant \|x_0\| + c$ for all $x \in E_c$, so E_c is bounded. It follows from the Arzela–Ascoli theorem that every sequence $(x_n)_{n\in\mathbb{N}}$ in E_c has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges (uniformly) to some $x \in C([0,1], \mathbb{R}^m)$. This subsequence also converges pointwise, so for every $t, s \in [0,1]$ we have

$$\|x(t) - x(s)\| = \lim_{n\to\infty} \underbrace{\|x_n(t) - x_n(s)\|}_{\leqslant c|t-s|} \leqslant c|t-s|.$$

Hence, $x \in X$.

(c) Since v is bounded, there is some $C > 0$ such that $0 \leqslant v(x) \leqslant C$ for every $x \in \mathbb{R}^m$. Hence, for every $x \in E_c$,

$$|f(x)| = \int_0^1 \underbrace{v(x(t))}_{\leqslant C} \underbrace{\left\| \tfrac{dx(t)}{dt} \right\|}_{\leqslant c} dt \leqslant \int_0^1 Cc = Cc.$$

We conclude that f is bounded.

(d) Let $g\colon \overline{E}_c \to \mathbb{R}$ be the lower semicontinuous envelope of f, which exists since f is bounded (Problem **2(a)**). Since g is lower semicontinuous (Problem **2(b)**) and $\overline{E}_c$ is compact (Problem **3(b)**), there exists a minimum for g, i.e., some $\bar{x} \in \overline{E}_c$ such that $g(\bar{x}) \leqslant g(y)$ for all $y \in \overline{E}_c$.

If $\bar{x} \in E_c$, then $f(\bar{x}) = g(\bar{x})$, by Problem **2(c)**, and for all $y \in E_c$, $f(\bar{x}) = g(\bar{x}) \leqslant g(y) = f(y)$. Hence, $\bar{x}$ is a minimum for f.

Remark. The proof that f is lower semicontinuous is quite technical, but here is an example that illustrates the point. Let $m = 2$, $x_0 = (0,0)$, $x_1 = (1,0)$ and

$k(x) = 1$ for all $x \in \mathbb{R}^2$. Define the functions $x, x_n \colon [0,1] \to \mathbb{R}^2$ by

$$x(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}, \qquad x_n(t) = \begin{pmatrix} t \\ \frac{\sin(n\pi t)}{n} \end{pmatrix}$$

for $n \in \mathbb{N}$ and $t \in [0,1]$. Then x and $x_1, x_2, \ldots$ lie in E_c for $c \geqslant 5$, and

$$\rho(x, x_n) = \sup_{t \in [0,1]} \frac{|\sin(n\pi t)|}{n} = \frac{1}{n},$$

so $x_n \to x$ as $n \to \infty$. However, $f(x) = 1$, while $f(x_n) \geqslant 2$ for all n, so f is not continuous at x (because that would imply $f(x_n) \to f(x)$). Still, $f(x) < f(y) + \varepsilon$ for $y = x_n$ for sufficiently large n. This indicates that f is lower semicontinuous at x (and in fact it is).