23 April 2026

MAT2400 – Real Analysis

Mandatory assignment 2 of 2

Submission deadline

Thursday 7 May 2026 at 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between writing in English or Norwegian. Use the Norwegian–English mathematical dictionary to look up terms.

You can choose between scanning handwritten notes or writing the solution directly on a computer with LaTeX. The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible; please use either the "Color" or "Grayscale" settings. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures.

Note that you have **one** attempt to pass the assignment. This means that there are no second attempts. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

Read the department's guidelines for mandatory assignments for more information.

IMPORTANT:
In order to get this mandatory assignment approved, **all you need to do is to submit a PDF file in Canvas.** It's up to you how many problems you submit. You will receive feedback on everything that you do submit.

You will find a list of relevant definitions on p. 8, and a list of hints on p. 9.

Problem 1. We will prove the following statement:

Every separable Hilbert space H has an orthonormal basis.

(a) Let $D = \{u_1, u_2, \ldots\}$ be a dense subset of H (why does such a set exist?). Show that there is a subset $D' \subseteq D$ with elements $D' = \{v_1, v_2, \ldots\}$, where $(v_1, v_2, \ldots)$ is linearly independent and $\operatorname{span}(v_1, v_2, \ldots)$ is dense in H.

(b) Apply the Gram–Schmidt orthogonalisation procedure to the list $(v_1, v_2, \ldots)$. Explain why the resulting list $(e_1, e_2, \ldots)$ is orthonormal and its span is dense in H.

(c) Prove that $(e_1, e_2, \ldots)$ is an orthonormal basis for H.

Solution:

(a) Let $n_1 \in \mathbb{N}$ be the smallest index for which u_{n_1} is nonzero. Let $n_2 > n_1$ be the smallest index for which the list (u_{n_1}, u_{n_2}) is linearly independent. Continue in this way, so that after k steps, the list $(u_{n_1}, \ldots, u_{n_k})$ is linearly independent and has the same span as $(u_1, u_2, \ldots, u_{n_k})$. Finally, denote $v_k \coloneqq u_{n_k}$ for $k \in \mathbb{N}$. Then $(v_1, v_2, \ldots)$ is linearly independent and has the same span as $(u_1, u_2, \ldots)$; in particular, the span contains D. The result now follows.

(b) After $n \in \mathbb{N}$ steps, the Gram–Schmidt procedure will have produced an orthonormal list $(e_1, \ldots, e_n)$ whose span equals that of $(v_1, \ldots, v_n)$. It follows that the span of the final list $(e_k)_{k\in\mathbb{N}}$ equals that of $(v_k)_{k\in\mathbb{N}}$, which is dense in H.

(c) Let $u \in H$. We claim that $u = \sum_{k=1}^{\infty} \langle u, e_k \rangle e_k$.

For $N \in \mathbb{N}$, denote $H_N \coloneqq \operatorname{span}(e_1, \ldots, e_N)$, and let $P_N \colon H \to H_N$ denote the orthogonal projection from H onto H_N. We know that

$$P_N u = \sum_{k=1}^{N} \langle u, e_k \rangle e_k,$$

and that $P_N u$ is the element in H_N that lies closest to u.

Since $E \coloneqq \operatorname{span}(e_k)_{k\in\mathbb{N}}$ is dense in H, there is for every $n \in \mathbb{N}$ some $u_n \in E$ such that $\|u - u_n\| < 1/n$. Let N_n be such that $u_n \in H_{N_n}$ for each $n \in \mathbb{N}$. Then

$$\|u - P_{N_n} u\| \leqslant \|u - u_n\| \leqslant 1/n,$$

so $P_{N_n} u \to u$ as $n \to \infty$. Since $H_1 \subseteq H_2 \subseteq \cdots$, we have $\|u - P_1 u\| \geqslant \|u - P_2 u\| \geqslant \cdots$, proving that $P_k u \to u$ as $k \to \infty$. By the explicit formula of P_k, we conclude that $u = \sum_{k=1}^{\infty} \langle u,\, e_k \rangle e_k$.

Finally, we claim that the coefficients $\langle u,\, e_k \rangle$ are unique. To this end, assume that u can be written as a convergent series $u = \sum_{k=1}^{\infty} \beta_k e_k$. Using continuity of the inner product to pull the series out from the inner product, we get for any $n \in \mathbb{N}$

$$\langle u,\, e_n \rangle = \Big\langle \sum_{k=1}^{\infty} \beta_k e_k,\, e_n \Big\rangle = \sum_{k=1}^{\infty} \beta_k \langle e_k,\, e_n \rangle = \beta_n.$$

Hence, $\beta_n = \langle u,\, e_n \rangle$ for all n, proving our claim.

Problem 2. We will prove the Riesz Representation Theorem:

> If H is a Hilbert space over $\mathbb{R}$ and $T \in \mathcal{L}(H, \mathbb{R})$, then there exists a unique vector $v \in H$ such that $Tu = \langle u,\, v \rangle$ for all $u \in H$.

(The same is true for Hilbert spaces over $\mathbb{C}$.) We will assume that H is separable, so that it has an orthonormal basis $(e_k)_{k \in \mathbb{N}}$.

(a) Show that there is some $C > 0$ such that

$$|Tu| \leqslant C \left(\sum_{k=1}^{\infty} |\langle u,\, e_k \rangle|^2 \right)^{1/2} \qquad \forall\, u \in H. \tag{1}$$

(b) Define $\alpha_k \coloneqq T e_k$ for $k \in \mathbb{N}$. Show that

$$\sum_{k=1}^{n} |\alpha_k|^2 \leqslant C \left(\sum_{k=1}^{n} |\alpha_k|^2 \right)^{1/2} \qquad \forall\, n \in \mathbb{N}. \tag{2}$$

Conclude that $\alpha \coloneqq (\alpha_1, \alpha_2, \dots) \in \ell^2(\mathbb{R})$ and that $\|\alpha\|_{\ell^2} \leqslant C$.

(c) Show that the series $v \coloneqq \sum_{k=1}^{\infty} \alpha_k e_k$ converges, and that

$$Tu = \langle u,\, v \rangle \qquad \forall\, u \in H.$$

Solution:

(a) Parseval's identity yields $\|u\| = \left(\sum_{k=1}^{\infty} |\langle u,\, e_k \rangle|^2 \right)^{1/2}$ for all $u \in H$. Thus,

$$|Tu| \leqslant C\|u\| = C \Big(\sum_{k=1}^{\infty} |\langle u,\, e_k \rangle|^2 \Big)^{1/2}.$$

(b) For some $n \in \mathbb{N}$, let $v_n \coloneqq \sum_{k=1}^n \alpha_k e_k$. Then

$$Tv_n = T\Big(\textstyle\sum_{k=1}^n \alpha_k e_k\Big) = \displaystyle\sum_{k=1}^n \alpha_k T e_k = \sum_{k=1}^n |\alpha_k|^2.$$

Combining this with the previous problem yields (2). Dividing by $\left(\sum_{k=1}^n |\alpha_k|^2\right)^{1/2}$ on both sides yields

$$\left(\sum_{k=1}^n |\alpha_k|^2\right)^{1/2} \leqslant C$$

for any $n \in \mathbb{N}$. Letting $n \to \infty$ yields

$$\|\alpha\|_{\ell^2} \coloneqq \left(\sum_{k=1}^\infty |\alpha_k|^2\right)^{1/2} \leqslant C,$$

so that $\alpha \in \ell^2(\mathbb{R})$.

(c) For every $n, m \in \mathbb{N}$ we have (assuming that, say, $m \leqslant n$)

$$\|v_n - v_m\|_{\ell^2}^2 = \sum_{k=m+1}^n |\alpha_k|^2 \leqslant \sum_{k=m+1}^\infty |\alpha_k|^2.$$

Since $\sum_{k=1}^\infty |\alpha_k|^2 < \infty$, there is for every $\varepsilon > 0$ some $N \in \mathbb{N}$ such that $\sum_{k=N}^\infty |\alpha_k|^2 < \varepsilon$. If $n, m \geqslant N$ then we can conclude that $\|v_n - v_m\|_{\ell^2} \leqslant \sqrt{\varepsilon}$. Hence, $(v_n)_{n\in\mathbb{N}}$ is Cauchy, and since H is complete, the sequence converges. Hence, the series v converges.

Let $u \in H$. If $u_n \coloneqq \sum_{k=1}^n \langle u, e_k\rangle e_k$ then $u_n \to u$ as $n \to \infty$, and

$$\begin{aligned} Tu_n &= \sum_{k=1}^n \langle u, e_k\rangle T e_k = \sum_{k=1}^n \langle u, e_k\rangle \alpha_k \\ &= \big\langle u, \textstyle\sum_{k=1}^n \alpha_k e_k\big\rangle = \langle u, v_n\rangle. \end{aligned}$$

Passing $n \to \infty$ and using the continuity of T and of $\langle u, \cdot\rangle$, we get

$$Tu = \lim_{n\to\infty} Tu_n = \lim_{n\to\infty} \langle u, v_n\rangle = \langle u, v\rangle,$$

which is what we wanted.

Finally, we prove that v is unique. If there is another $w \in H$ such that $Tu = \langle u, w\rangle$ for all $u \in H$, then for all $u \in H$

$$0 = Tu - Tu = \langle u, v\rangle - \langle u, w\rangle = \langle u, v - w\rangle,$$

and setting $u = v - w$ gives $\|v - w\|^2 = 0$, whence $v = w$.

Problem 3. Let X be a set and let $\mathcal{S} \coloneqq \big\{\{x\} : x \in X\big\}$ (the set of *singletons* in X). Let $\mathcal{A} \coloneqq \sigma(\mathcal{S})$ be the σ-algebra generated by $\mathcal{S}$.

(a) Show that

$$\mathcal{A} = \big\{E \subseteq X : E \text{ or } E^c \text{ is countable}\big\}.$$

(Here, $E^c \coloneqq X \setminus E$, and by "countable" we mean either finite or countably infinite.)

(b) Assume that X is uncountable. Show that there exists a set $E \subseteq X$ which is not measurable with respect to $\mathcal{A}$.

Solution:

(a) Denote $\mathcal{B} \coloneqq \big\{E \subseteq X : E \text{ or } E^c \text{ is countable}\big\}$. By taking unions of sets in $\mathcal{S}$, we can obtain any countable set $E \subseteq X$, and by taking their complement, any set whose complement is countable. This proves that $\mathcal{A} \supseteq \mathcal{B}$. It remains to prove that $\mathcal{B}$ is a σ-algebra. This will imply that $\mathcal{A} \subseteq \mathcal{B}$, because $\mathcal{A}$ is the smallest σ-algebra containing $\mathcal{S}$.

The set $\emptyset$ is countable, and hence in $\mathcal{B}$. It is also clear that if $E \in \mathcal{B}$, then $E^c \in \mathcal{B}$. Finally, let $E_1, E_2, \ldots \in \mathcal{B}$. If all of these sets are countable, then their union is countable, and therefore in $\mathcal{B}$. If at least one of the sets is uncountable, let $I \subseteq \mathbb{N}$ be the indices of those sets that are countable, so that E_k is countable for all $k \in I$, and E_k^c is countable for all $k \in \mathbb{N} \setminus I$. Then

$$\bigcup_{k\in\mathbb{N}} E_k = \bigcup_{k\in I} E_k \cup \bigcup_{k\in\mathbb{N}\setminus I} E_k = \Bigg(\underbrace{\bigcap_{k\in I} E_k^c \cap \bigcap_{k\in\mathbb{N}\setminus I} E_k^c}_{=:\,F}\Bigg)^c.$$

The set F is countable, since it is the intersection between at least one countable set and other sets; hence, the above is the complement of a countable set, and therefore lies in $\mathcal{B}$. This proves the claim.

(b) Let $E \subseteq X$ be a set such that neither E nor E^c is countable. Then E clearly does not lie in $\mathcal{A}$, and hence is not measurable.

To see that such a set E exists requires some basic set theory, but here is a sketch. If $|X|$ denotes the cardinality of X, and $|X| + |X|$ denotes the cardinality of $X \times \{1, 2\}$, then it is a fact of cardinal arithmetic that $|X| + |X| = |X|$ whenever X is infinite (see for instance Halmos' *Naive set theory*, pp. 96–97). Hence, if $X_1 \coloneqq X \times \{1\}$ and $X_2 \coloneqq X \times \{2\}$, then there exists a bijection $f \colon X_1 \cup X_2 \to X$. Let $E \coloneqq f(X_1)$. Then E, being the image under a bijection of an uncountable set, is uncountable, while $E^c = f(X_2)$ is also uncountable.

Remark. In *probability theory* one works with measure spaces $(X, \mathcal{A}, \mu)$ where μ is a *probability measure*, meaning that μ satisfies $\mu(X) = 1$. The elements $x \in X$ are called *outcomes*. A measurable set $E \in \mathcal{A}$ is called an *event*, and the number $\mu(E)$ (which must then be a number between 0 and 1) is interpreted as the probability of the event E.

Problem 4. Let $X = \{x_1, \dots, x_n\}$ be a set containing $n \in \mathbb{N}$ elements, and equip it with the discrete σ-algebra $\mathcal{A} \coloneqq 2^X$. Let $p_1, \dots, p_n \in [0,1]$ be such that $\sum_{k=1}^n p_k = 1$. Show that there exists a unique probability measure μ on $(X, \mathcal{A})$ with the property that $\mu(\{x_k\}) = p_k$ for $k = 1, \dots, n$.

Solution: Define

$$\mu(E) \coloneqq \sum_{\substack{k \in \{1,\dots,n\} \\ x_k \in E}} p_k \qquad \forall\, E \subseteq X.$$

Then $\mu(E) \in [0,1]$ for every $E \in \mathcal{A}$, and $\mu(\emptyset) = 0$ and $\mu(X) = \sum_{k=1}^n p_k = 1$. If $E_1, E_2, \dots \in \mathcal{A}$ are disjoint then, since X contains only n elements, at most n of the sets can be nonempty. Reordering the sets so that $E_j = \emptyset$ for all $j > n$, we get

$$\begin{aligned}
\mu\left(\bigcup_{j\in\mathbb{N}} E_j\right) = \mu\left(\bigcup_{j=1,\dots,n} E_j\right) &= \sum_{k\,:\,x_k \in E_1 \cup \cdots \cup E_n} p_k \\
&= \sum_{j=1}^n \left(\sum_{k\,:\,x_k \in E_j} p_k \right) \\
&= \sum_{j=1}^n \mu(E_j) \\
&= \sum_{j=1}^\infty \mu(E_j),
\end{aligned}$$

where we in the third step have used the fact that $E_1, \dots, E_n$ are disjoint, which implies that each x_k can lie in at most one set E_j. This proves that μ is a measure.

If ρ is any other measure such that $\rho(\{x_k\}) = p_k$ for each k, then for any $E \in \mathcal{A}$,

$$\begin{aligned}
\rho(E) = \rho\Bigg(\underbrace{\bigcup_{k\,:\,x_k \in E} \{x_k\}}_{\text{disjoint, finite union}} \Bigg) &= \sum_{k\,:\,x_k \in E} \rho(\{x_k\}) \\
&= \sum_{k\,:\,x_k \in E} p_k = \sum_{k\,:\,x_k \in E} \mu(\{x_k\}) \\
&= \mu\left(\bigcup_{k\,:\,x_k \in E} \{x_k\} \right) = \mu(E).
\end{aligned}$$

Hence, $\rho = \mu$, so the measure is unique.

Problem 5. We toss a coin three times, and record the result as a vector (a_1, a_2, a_3), where $a_1, a_2, a_3 \in \{0, 1\}$, and where 0 encodes "heads" and 1 encodes "tails". Let

$$X \coloneqq \big\{(a_1, a_2, a_3) : a_1, a_2, a_3 \in \{0,1\}\big\},$$

and let $\mathcal{A} \coloneqq 2^X$.

(a) If heads and tails have equal likelihoods, what is the correct probability measure μ on the measurable space $(X, \mathcal{A})$?

(b) Find the set $E \in \mathcal{A}$ which describes the event "at least two tosses were tails".

(c) Two events $E, F \in \mathcal{A}$ are *independent* if the probability of both occurring equals the product of the probability of each of them:

$$\mu(E \cap F) = \mu(E)\mu(F).$$

Show that the events "the first toss is heads" and "the first and second tosses are the same" are independent.

Solution:

(a) The probability of any outcome (a_1, a_2, a_3) is $(1/2)^3 = 1/8$, so the correct probability measure is the one which assigns $\mu(\{a\}) = 1/8$ to each outcome $a \in X$.

(b) The event "at least two tosses were tails" is described by the outcomes

$$(1,1,0),\ (1,0,1),\ (0,1,1),\ (1,1,1).$$

Hence, the event E is the set

$$E = \big\{(1,1,0),\ (1,0,1),\ (0,1,1),\ (1,1,1)\big\}.$$

(c) Let E be the event "the first toss is heads", and let F be the event "the first and second tosses are the same". Then

$$E = \big\{(0,0,0),\ (0,0,1),\ (0,1,0),\ (0,1,1)\big\},$$
$$F = \big\{(0,0,0),\ (0,0,1),\ (1,1,0),\ (1,1,1)\big\}.$$

Since E and F contain 4 elements each, we have $\mu(E) = \mu(F) = 4\frac{1}{8} = \frac{1}{2}$. Next,

$$E \cap F = \big\{(0,0,0),\ (0,0,1)\big\},$$

so $\mu(E \cap F) = 2\frac{1}{8} = \frac{1}{4}$. This coincides with $\mu(E)\mu(F) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$. Hence, E and F are independent.

Problem 6. Let $(X, \mathcal{A}, \mu)$ be a measure space. For a list of measurable sets $E_1, E_2, \ldots \in \mathcal{A}$, we define the *limit superior*

$$\limsup_{n\to\infty} E_n \coloneqq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k.$$

(a) Prove that $\limsup_{n\to\infty} E_n$ is measurable.

(b) Prove that

$$\limsup_{n\to\infty} E_n = \big\{x \in X : x \text{ lies in infinitely many of the sets } E_1, E_2, \dots\big\}.$$

(c) Prove the Borel–Cantelli lemma:

If $\sum_{k=1}^{\infty} \mu(E_k) < \infty$, then $\mu\big(\limsup_{n\to\infty} E_n\big) = 0$.

Solution:

(a) Each set E_k is measurable, so the union $\cup_{k\geqslant n} E_k$ is measurable, and therefore the intersection $\cap_{n\in\mathbb{N}}\Big(\cup_{k\geqslant n} E_k\Big)$ is measurable.

(b) We have $x \in \limsup_{n\to\infty} E_n$ if and only if $x \in \cup_{k\geqslant n} E_k$ for all $n \in \mathbb{N}$, which is the same as saying that for every $n \in \mathbb{N}$, there is some $k \geqslant n$ such that $x \in E_k$. This is the same as saying that there are infinitely many k for which $x \in E_k$. In other words, x lies in infinitely many of the sets $E_1, E_2, \dots$.

(c) Let $\varepsilon > 0$. Since $\sum_k \mu(E_k) < \infty$, there is some $N \in \mathbb{N}$ such that $\sum_{k=N}^{\infty} \mu(E_k) < \varepsilon$. We have

$$\begin{aligned}\limsup_{n\to\infty} E_n &\coloneqq \bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k=n}^{\infty} E_k}_{\subseteq \cup_{k\geqslant N} E_k \text{ for } n\geqslant N} \\ &\subseteq \bigcup_{k=N}^{\infty} E_k,\end{aligned}$$

so

$$\mu\Big(\limsup_{n\to\infty} E_n\Big) \leqslant \mu\left(\bigcup_{k=N}^{\infty} E_k\right) \leqslant \sum_{k=N}^{\infty} \mu(E_k) < \varepsilon.$$

Since ε was arbitrary, we conclude that $\mu\Big(\limsup_{n\to\infty} E_n\Big) = 0$.

Relevant definitions

Definition. A *Hilbert space* is a vector space H equipped with an inner product $\langle \cdot, \cdot \rangle$ such that if $\|u\| \coloneqq \sqrt{\langle u, u \rangle}$ is the induced norm, then $(H, \|\cdot\|)$ is a complete normed vector space.

Definition. A normed vector space $(U, \|\cdot\|)$ is *separable* if it has a dense, countable subset — that is, there exists a countable subset $D \subseteq U$ such that for any $u \in U$ and $\varepsilon > 0$, there exists some $v \in D$ such that $\|u - v\| < \varepsilon$.

Definition. In a vector space U over a field $\mathbb{K}$ and for vectors $u_1, u_2, \ldots \in U$, the *span* of the list $(u_1, u_2, \ldots)$ is the set

$$\begin{aligned} \operatorname{span}(u_1, u_2, \ldots) &\coloneqq \{\text{all finite linear combinations of } u_1, u_2, \ldots\} \\ &= \left\{ \sum_{k=1}^{N} \alpha_k u_k : N \in \mathbb{N},\ \alpha_1, \ldots, \alpha_N \in \mathbb{K} \right\}. \end{aligned}$$

Definition. In a finite-dimensional vector space U, a *basis* is a finite list $(e_1, \ldots, e_n)$ of vectors in U such that any $u \in U$ can be written as $u = \sum_{k=1}^{n} \alpha_k e_k$ for unique scalars $\alpha_1, \ldots, \alpha_n$.

In an infinite-dimensional normed vector space $(U, \|\cdot\|)$, a *basis* is an infinite list $(e_1, e_2, \ldots)$ of vectors in U such that any $u \in U$ can be written as $u = \sum_{k=1}^{\infty} \alpha_k e_k$ for unique scalars $\alpha_1, \alpha_2, \ldots$. (The series $\sum_{k=1}^{\infty} \alpha_k e_k$ should be interpreted as the limit as $n \to \infty$ of $\sum_{k=1}^{n} \alpha_k e_k$.)

Definition. In a vector space U with an inner product $\langle \cdot, \cdot \rangle$, a list $(e_k)_{k \in I}$ (where I is either a finite or countably infinite index set) is *orthonormal* if for any $j, k \in I$,

$$\langle e_j, e_k \rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{else.} \end{cases}$$

Definition. For normed vector spaces $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_V)$, a function $T \colon U \to V$ is a *bounded linear operator* (or *continuous linear operator*) if T is linear, and if there exists some $C > 0$ such that $\|Tu\|_V \leqslant C\|u\|_U$ for all $u \in U$.

Definition. The set $\ell^2(\mathbb{R})$ consists of all infinitely long vectors $\alpha = (\alpha_1, \alpha_2, \ldots)$ with components $\alpha_1, \alpha_2, \ldots \in \mathbb{R}$, such that $\|\alpha\|_{\ell^2} < \infty$, where

$$\|\alpha\|_{\ell^2} \coloneqq \left(\sum_{k=1}^{\infty} |\alpha_k|^2 \right)^{1/2}.$$

Hints

1(c): Let P_N be the orthogonal projection onto the subspace $H_N \coloneqq \operatorname{span}(e_1, \ldots, e_N)$. Prove that $P_N u \to u$ for every $u \in H$.

2(a): Use Parseval's identity (Theorem 5.3.10): For every $u \in H$ we have $u = \sum_{k=1}^{\infty} \langle u,\, e_k \rangle e_k$, and $\|u\| = \left(\sum_{k=1}^{\infty} |\langle u,\, e_k \rangle|^2\right)^{1/2}$.

2(b): Define $v_n \coloneqq \sum_{k=1}^{n} \alpha_k e_k$ and use (a) to estimate $|Tv_n|$.

3(a): Let $\mathcal{B} \coloneqq \big\{E \subseteq X : E \text{ or } E^c \text{ is countable}\big\}$. It is enough to prove that

1. $\mathcal{B} \subseteq \mathcal{A}$
2. $\mathcal{B}$ is a σ-algebra.

(You need to explain why this is enough.)

6(c): For an $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that $\sum_{k=N}^{\infty} \mu(E_k) < \varepsilon$.