23 April 2026

MAT2400 – Real Analysis

Mandatory assignment 2 of 2

Submission deadline

Thursday 7 May 2026 at 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between writing in English or Norwegian. Use the Norwegian–English mathematical dictionary to look up terms.

You can choose between scanning handwritten notes or writing the solution directly on a computer with LaTeX. The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible; please use either the "Color" or "Grayscale" settings. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures.

Note that you have **one** attempt to pass the assignment. This means that there are no second attempts. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

Read the department's guidelines for mandatory assignments for more information.

IMPORTANT:
In order to get this mandatory assignment approved, **all you need to do is to submit a PDF file in Canvas.** It's up to you how many problems you submit. You will receive feedback on everything that you do submit.

You will find a list of relevant definitions on p. 4, and a list of hints on p. 5.

Problem 1. We will prove the following statement:

> Every separable Hilbert space H has an orthonormal basis.

(a) Let $D = \{u_1, u_2, \dots\}$ be a dense subset of H (why does such a set exist?). Show that there is a subset $D' \subseteq D$ with elements $D' = \{v_1, v_2, \dots\}$, where $(v_1, v_2, \dots)$ is linearly independent and $\operatorname{span}(v_1, v_2, \dots)$ is dense in H.

(b) Apply the Gram–Schmidt orthogonalisation procedure to the list $(v_1, v_2, \dots)$. Explain why the resulting list $(e_1, e_2, \dots)$ is orthonormal and its span is dense in H.

(c) Prove that $(e_1, e_2, \dots)$ is an orthonormal basis for H.

Problem 2. We will prove the Riesz Representation Theorem:

> If H is a Hilbert space over $\mathbb{R}$ and $T \in \mathcal{L}(H, \mathbb{R})$, then there exists a unique vector $v \in H$ such that $Tu = \langle u, v\rangle$ for all $u \in H$.

(The same is true for Hilbert spaces over $\mathbb{C}$.) We will assume that H is separable, so that it has an orthonormal basis $(e_k)_{k\in\mathbb{N}}$.

(a) Show that there is some $C > 0$ such that

$$|Tu| \leqslant C\left(\sum_{k=1}^{\infty} |\langle u, e_k\rangle|^2\right)^{1/2} \qquad \forall\, u \in U. \tag{1}$$

(b) Define $\alpha_k \coloneqq Te_k$ for $k \in \mathbb{N}$. Show that

$$\sum_{k=1}^{n} |\alpha_k|^2 \leqslant C\left(\sum_{k=1}^{n} |\alpha_k|^2\right)^{1/2} \qquad \forall\, n \in \mathbb{N}. \tag{2}$$

Conclude that $\alpha \coloneqq (\alpha_1, \alpha_2, \dots) \in \ell^2(\mathbb{R})$ and that $\|\alpha\|_{\ell^2} \leqslant C$.

(c) Show that the series $v \coloneqq \sum_{k=1}^{\infty} \alpha_k e_k$ converges, and that

$$Tu = \langle u, v\rangle \qquad \forall\, u \in H.$$

Problem 3. Let X be a set and let $\mathcal{S} \coloneqq \big\{\{x\} : x \in X\big\}$ (the set of *singletons* in X). Let $\mathcal{A} \coloneqq \sigma(\mathcal{S})$ be the σ-algebra generated by $\mathcal{S}$.

(a) Show that

$$\mathcal{A} = \big\{E \subseteq X : E \text{ or } E^c \text{ is countable}\big\}.$$

(Here, $E^c \coloneqq X \setminus E$, and by "countable" we mean either finite or countably infinite.)

(b) Assume that X is uncountable. Show that there exists a set $E \subseteq X$ which is not measurable with respect to $\mathcal{A}$.

Remark. In *probability theory* one works with measure spaces $(X, \mathcal{A}, \mu)$ where μ is a *probability measure*, meaning that μ satisfies $\mu(X) = 1$. The elements $x \in X$ are called *outcomes*. A measurable set $E \in \mathcal{A}$ is called an *event*, and the number $\mu(E)$ (which must then be a number between 0 and 1) is interpreted as the probability of the event E.

Problem 4. Let $X = \{x_1, \dots, x_n\}$ be a set containing $n \in \mathbb{N}$ elements, and equip it with the discrete σ-algebra $\mathcal{A} \coloneqq 2^X$. Let $p_1, \dots, p_n \in [0, 1]$ be such that $\sum_{k=1}^n p_k = 1$. Show that there exists a unique probability measure μ on $(X, \mathcal{A})$ with the property that $\mu(\{x_k\}) = p_k$ for $k = 1, \dots, n$.

Problem 5. We toss a coin three times, and record the result as a vector (a_1, a_2, a_3), where $a_1, a_2, a_3 \in \{0, 1\}$, and where 0 encodes "heads" and 1 encodes "tails". Let

$$X \coloneqq \big\{(a_1, a_2, a_3) : a_1, a_2, a_3 \in \{0, 1\}\big\},$$

and let $\mathcal{A} \coloneqq 2^X$.

(a) If heads and tails have equal likelihoods, what is the correct probability measure μ on the measurable space $(X, \mathcal{A})$?

(b) Find the set $E \in \mathcal{A}$ which describes the event "at least two tosses were tails".

(c) Two events $E, F \in \mathcal{A}$ are *independent* if the probability of both occurring equals the product of the probability of each of them:

$$\mu(E \cap F) = \mu(E)\mu(F).$$

Show that the events "the first toss is heads" and "the first and second tosses are the same" are independent.

Problem 6. Let $(X, \mathcal{A}, \mu)$ be a measure space. For a list of measurable sets $E_1, E_2, \dots \in \mathcal{A}$, we define the *limit superior*

$$\limsup_{n \to \infty} E_n \coloneqq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k.$$

(a) Prove that $\limsup_{n\to\infty} E_n$ is measurable.

(b) Prove that

$$\limsup_{n\to\infty} E_n = \big\{x \in X : x \text{ lies in infinitely many of the sets } E_1, E_2, \dots\big\}.$$

(c) Prove the Borel–Cantelli lemma:

If $\sum_{k=1}^{\infty} \mu(E_k) < \infty$, then $\mu\big(\limsup_{n\to\infty} E_n\big) = 0$.

Relevant definitions

Definition. A *Hilbert space* is a vector space H equipped with an inner product $\langle \cdot, \cdot \rangle$ such that if $\|u\| \coloneqq \sqrt{\langle u, u \rangle}$ is the induced norm, then $(H, \|\cdot\|)$ is a complete normed vector space.

Definition. A normed vector space $(U, \|\cdot\|)$ is *separable* if it has a dense, countable subset — that is, there exists a countable subset $D \subseteq U$ such that for any $u \in U$ and $\varepsilon > 0$, there exists some $v \in D$ such that $\|u - v\| < \varepsilon$.

Definition. In a vector space U over a field $\mathbb{K}$ and for vectors $u_1, u_2, \ldots \in U$, the *span* of the list $(u_1, u_2, \ldots)$ is the set

$$\begin{aligned} \operatorname{span}(u_1, u_2, \ldots) &\coloneqq \{\text{all finite linear combinations of } u_1, u_2, \ldots\} \\ &= \left\{ \sum_{k=1}^{N} \alpha_k u_k : N \in \mathbb{N},\ \alpha_1, \ldots, \alpha_N \in \mathbb{K} \right\}. \end{aligned}$$

Definition. In a finite-dimensional vector space U, a *basis* is a finite list $(e_1, \ldots, e_n)$ of vectors in U such that any $u \in U$ can be written as $u = \sum_{k=1}^{n} \alpha_k e_k$ for unique scalars $\alpha_1, \ldots, \alpha_n$.

In an infinite-dimensional normed vector space $(U, \|\cdot\|)$, a *basis* is an infinite list $(e_1, e_2, \ldots)$ of vectors in U such that any $u \in U$ can be written as $u = \sum_{k=1}^{\infty} \alpha_k e_k$ for unique scalars $\alpha_1, \alpha_2, \ldots$. (The series $\sum_{k=1}^{\infty} \alpha_k e_k$ should be interpreted as the limit as $n \to \infty$ of $\sum_{k=1}^{n} \alpha_k e_k$.)

Definition. In a vector space U with an inner product $\langle \cdot, \cdot \rangle$, a list $(e_k)_{k \in I}$ (where I is either a finite or countably infinite index set) is *orthonormal* if for any $j, k \in I$,

$$\langle e_j, e_k \rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{else.} \end{cases}$$

Definition. For normed vector spaces $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_V)$, a function $T \colon U \to V$ is a *bounded linear operator* (or *continuous linear operator*) if T is linear, and if there exists some $C > 0$ such that $\|Tu\|_V \leqslant C\|u\|_U$ for all $u \in U$.

Definition. The set $\ell^2(\mathbb{R})$ consists of all infinitely long vectors $\alpha = (\alpha_1, \alpha_2, \ldots)$ with components $\alpha_1, \alpha_2, \ldots \in \mathbb{R}$, such that $\|\alpha\|_{\ell^2} < \infty$, where

$$\|\alpha\|_{\ell^2} \coloneqq \left(\sum_{k=1}^{\infty} |\alpha_k|^2 \right)^{1/2}.$$

Hints

1(c): Let P_N be the orthogonal projection onto the subspace $H_N \coloneqq \operatorname{span}(e_1, \ldots, e_N)$. Prove that $P_N u \to u$ for every $u \in H$.

2(a): Use Parseval's identity (Theorem 5.3.10): For every $u \in H$ we have $u = \sum_{k=1}^{\infty} \langle u,\, e_k \rangle e_k$, and $\|u\| = \left(\sum_{k=1}^{\infty} |\langle u,\, e_k \rangle|^2\right)^{1/2}$.

2(b): Define $v_n \coloneqq \sum_{k=1}^{n} \alpha_k e_k$ and use (a) to estimate $|Tv_n|$.

3(a): Let $\mathcal{B} \coloneqq \big\{E \subseteq X : E \text{ or } E^c \text{ is countable}\big\}$. It is enough to prove that

1. $\mathcal{B} \subseteq \mathcal{A}$
2. $\mathcal{B}$ is a σ-algebra.

(You need to explain why this is enough.)

6(c): For an $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that $\sum_{k=N}^{\infty} \mu(E_k) < \varepsilon$.