Problems on the p-norm

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For $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ and $p \in [1, \infty)$, we define the p-norm

$$\|u\|_p := \left(\sum_{k=1}^{n} |u_k|^p \right)^{1/p},$$

and for "$p = \infty$", we define

$$\|u\|_\infty := \max_{k=1,\dots,n} |u_k|.$$

Our goal is to show that these are all norms on $\mathbb{R}^n$. Recall that a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is a *norm* on $\mathbb{R}^n$ if

(i) (*Positivity*) $\|u\| \geqslant 0$ for all $u \in \mathbb{R}^n$, and $\|u\| = 0$ only when $u = 0$ (the zero vector).
(ii) (*Homogeneity*) $\|\alpha u\| = |\alpha| \|u\|$ for all $\alpha \in \mathbb{R}$ and $u \in \mathbb{R}^n$.
(iii) (*The triangle inequality*) $\|u + v\| \leqslant \|u\| + \|v\|$ for all $u, v \in \mathbb{R}^n$.

Problem 1. Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms on $\mathbb{R}^n$.

For the remaining problems we assume that $p \in (1, \infty)$.

Problem 2. Show that $\|\cdot\|_p$ satisfies the positivity and homogeneity conditions.

As usual, the triangle inequality is the hardest thing to prove, and that requires several tricks.

Definition 3. We say that $p, q \in (1, \infty)$ are *conjugate exponents* if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We say that the conjugate exponent of 1 is ∞, and vice versa.

Problem 4. Let p, q be conjugate exponents.

(**a**) Show that if $p = q$, then $p, q = 2$.

(**b**) Show that if $p < 2$ then $q > 2$ (and vice versa).

(**c**) Show that $q = \frac{p}{p-1}$.

Problem 5. Prove *Young's inequality*: If $p, q \in (1, \infty)$ are conjugate exponents, then

$$xy \leqslant \frac{|x|^p}{p} + \frac{|y|^q}{q} \qquad \forall\, x, y \in \mathbb{R}. \tag{1}$$

You may proceed as follows:

(**a**) For a continuous function $f : \mathbb{R} \to \mathbb{R}$, the *Legendre transform* of f is the function $f^* : \mathbb{R} \to \mathbb{R}$ defined as

$$f^*(y) := \sup_{z \in \mathbb{R}} \big(zy - f(z)\big).$$

Prove that

$$xy \leqslant f(x) + f^*(y) \qquad \forall\, x, y \in \mathbb{R}.$$

(**b**) Let $p, q \in (1, \infty)$ be conjugate exponents. Show that the Legendre transform of $f(x) := |x|^p/p$ is $f^*(y) = |y|^q/q$.
Hint: Think of $f^*(y)$ as the maximum of a function. How do you find maxima of functions on $\mathbb{R}$?

(**c**) Prove Young's inequality (1).

Problem 6. Next, we prove *Hölder's inequality*: If $u, v \in \mathbb{R}^n$, and $p, q \in [1, \infty]$ are conjugate exponents, then

$$\|uv\|_1 \leqslant \|u\|_p \|v\|_q. \tag{2}$$

(Here, uv is the vector $uv = (u_1 v_1, \ldots, u_n v_n)$.)

(**a**) Prove the special case $p = 1, q = \infty$.

(**b**) We now assume $p, q \in (1, \infty)$. Prove Hölder's inequality in the case where either $u = 0$ or $v = 0$.

(**c**) Assume now that $u, v \neq 0$. Explain each step in the following computation:

$$\begin{aligned}
\frac{\|uv\|_1}{\|u\|_p \|v\|_q} &= \frac{1}{\|u\|_p \|v\|_q} \sum_{i=1}^n |u_i v_i| = \sum_{i=1}^n \frac{|u_i|}{\|u\|_p} \frac{|v_i|}{\|v\|_q} \\
&\leqslant \sum_{i=1}^n \frac{|u_i|^p}{p\|u\|_p^p} + \frac{|v_i|^q}{q\|v\|_q^q} \\
&= \frac{\|u\|_p^p}{p\|u\|_p^p} + \frac{\|v\|_q^q}{q\|v\|_q^q} = 1.
\end{aligned}$$

(**d**) Use the computation in the previous step to conclude Hölder's inequality.

Problem 7. We can now finally prove the triangle inequality for the p-norm, for $p \in (1, \infty)$. Let $u, v \in \mathbb{R}^n$.

(**a**) Prove the triangle inequality in the special case where $u + v = 0$.

(**b**) Assume now $u + v \neq 0$. Explain each step in the following computation:

$$\begin{aligned}
\|u+v\|_p^p &= \sum_{i=1}^n |u_i+v_i|^p = \sum_{i=1}^n |u_i+v_i| \cdot |u_i+v_i|^{p-1} \\
&\leqslant \sum_{i=1}^n |u_i| \cdot |u_i+v_i|^{p-1} + \sum_{i=1}^n |v_i| \cdot |u_i+v_i|^{p-1} \\
&\leqslant \left(\sum_{i=1}^n |u_i|^p\right)^{1/p} \left(\sum_{i=1}^n |u_i+v_i|^{(p-1)q}\right)^{1/q} \\
&\quad + \left(\sum_{i=1}^n |v_i|^p\right)^{1/p} \left(\sum_{i=1}^n |u_i+v_i|^{(p-1)q}\right)^{1/q} \\
&= \|u\|_p \|u+v\|_p^{p-1} + \|v\|_p \|u+v\|_p^{p-1}.
\end{aligned}$$

(**c**) Use the computation in the previous step to conclude the triangle inequality.