Defn A complex number is an expression of the form

$$z = x + iy$$

where $x, y \in \mathbb{R}$.

$$x = \operatorname{Re} z \qquad (\text{the real part of } z)$$
$$y = \operatorname{Im} z \qquad (\text{the imaginary part of } z)$$

$\mathbb{C}$ = the set of complex numbers $= \{ z = x+iy \mid x, y \in \mathbb{R} \}$.

What is i? To define $\mathbb{C}$ properly we should define it via $\mathbb{R}^2$:

$$\mathbb{C} \longleftrightarrow \mathbb{R}^2$$
$$z = x+iy \longleftrightarrow (x,y)$$

← we know how to add vectors in $\mathbb{R}^2$.

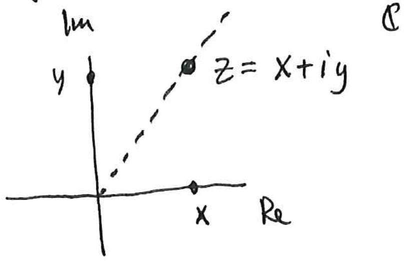
And define a multiplication on $\mathbb{R}^2$ via the rules

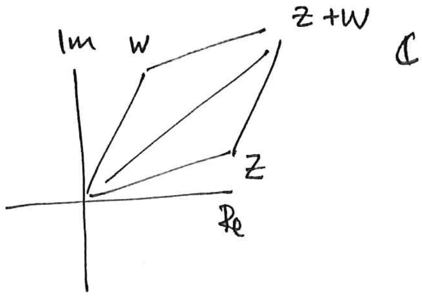
$$(a,b) + (c,d) = (a+c, b+d)$$
$$(a,b)\cdot(c,d) = (ac-bd, ad+bc)$$

↑ This makes $\mathbb{C}$ into a field: You can add, subtract, multiply elements and inverses of non-zero elements exist.

Then i is simply the element corresponding to $(0,1)$.
and 1 corresponds to $(1,0)$).

Graphical representation:





← the picture is of $\mathbb{R}^2$ but we identify it with $\mathbb{C}$.

$|z| = \sqrt{x^2+y^2}$ (the "modulus" of z, or the "absolute value" or "norm")

This satisfies the triangle inequality:

$$|z+w| \le |z| + |w| \qquad \forall z, w \in \mathbb{C}$$

Useful version of this inequality (apply it to $z' = z$ and $w' = z - w$):

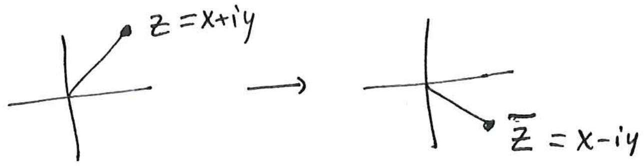
$$|z-w| \geq |z| - |w|$$

Defn If $z = x+iy$ is a complex number, the complex conjugate of z is

$$\bar{z} = x - iy$$

$\overline{(\bar{z})} = z$.

Geometrically :



$\therefore$ reflection in the x-axis

We have the following properties :

$$\overline{z+w} = \bar{z} + \bar{w}$$
$$\overline{zw} = \bar{z}\,\bar{w}$$
$$|z| = |\bar{z}|$$
$$|z|^2 = z\bar{z} \rightsquigarrow \frac{1}{z} = \frac{\bar{z}}{|z|^2} \quad \text{and} \quad \begin{aligned} |zw| &= (zw)(\overline{zw}) \\ &= (z\bar{z})(w\bar{w}) \\ &= |z||w| \end{aligned}$$

If $z = x+iy$ then

$$x = \operatorname{Re} z = \frac{z + \bar{z}}{2}$$
$$y = \operatorname{Im} z = \frac{z - \bar{z}}{2i}$$

<u>Defn</u> A complex polynomial of degree $n \geq 0$ is a function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 \qquad a_i \in \mathbb{C}.$$

<u>The Fundamental Theorem of Algebra</u> : Every complex polynomial $p(z)$ has a factorization

$$p(z) = c \cdot (z - z_1)^{m_1} \cdots (z - z_k)^{m_k}$$

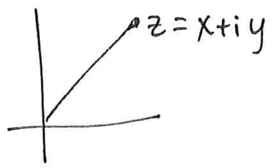
where $z_i \neq z_j$ $\forall i,j$ and $m_i \geq 1$. This factorization is unique up to permutation of the factors.

The hard part is showing that $p(z)$ has <u>at least one zero</u> (root). From there the theorem is proved using induction on the degree.

Uniqueness is clear: z_i are the roots of p, m_i = integer s.t $\frac{p(z) - p(z_1)}{z - z_1} =$. $p(x) = (z - z_i)^{m_i} q(z$ and $q(z_i) \neq$.

Existance : Given one z_1 st $p(z_1) = 0$ $\rightsquigarrow$ $p(z) = (z - z_1) \cdot q(z)$ for some polynomial $q(z)$. By induction, we can factor $q(z)$ $\to$ can factor

Polar representation



$r = \sqrt{x^2+y^2} \Rightarrow \begin{array}{l} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \end{array}$ for some θ

$\Rightarrow z = r(\cos\theta + i\sin\theta)$

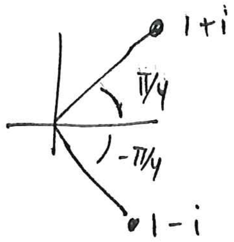
Defn The argument of z is $\arg z = \theta$ (multivalued)
The principal value of $\arg z$ is $\operatorname{Arg} z = \theta \mod [-\pi,\pi] \in [-\pi,\pi]$.

$\therefore \arg z = \{\operatorname{Arg} z + 2\pi k \mid k \in \mathbb{Z}\}$

Ex

$\operatorname{Arg}(1+i) = \pi/4$

$\operatorname{Arg}(1-i) = -\pi/4$



$\therefore$ More generally $\operatorname{Arg}\bar{z} = -\operatorname{Arg} z$.

$z = -2-3i$

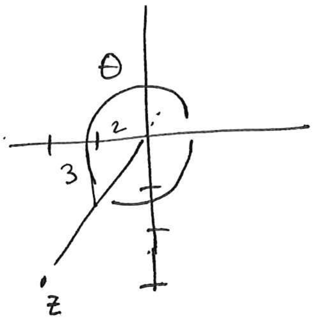
$\tan\left(\frac{-3}{-2}\right) \approx 0.9828$

Note: $\theta \neq \operatorname{Arg}(z)$ since

$\theta \approx 0.9828 + \pi \approx 4.124$

$\rightsquigarrow$ subtract 2π

$\rightsquigarrow \operatorname{Arg} \approx -2.159$



$\therefore \operatorname{Arg} z = \tan^{-1}\left(\frac{3}{2}\right) - \pi$

$\arg z = \tan^{-1}\left(\frac{3}{2}\right) - \pi + 2\pi k \quad k \in \mathbb{Z}$.

Defn The imaginary exponential is defined by

$e^{i\theta} = \cos\theta + i\sin\theta \qquad \underline{\theta \in \mathbb{R}}$

$z \neq 0$, then $z = |z|e^{i\theta}$ is called the polar form of z.
or polar representation

Note:

$e^{i(\theta+2\pi k)} = e^{i\theta}$

$|e^{i\theta}| = 1$

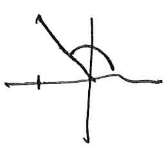
$\frac{1}{e^{i\theta}} = e^{-i\theta}$

$e^{i(\theta+\theta')} = e^{i\theta}\cdot e^{i\theta'}$ (this is equivalent to $\cos(\theta+\theta') + i\sin(\theta+\theta')$
$= (\cos\theta + i\sin\theta)(\cos\theta' + i\sin\theta')$
$= \cdots$

ex

$z = -1+i \Rightarrow |z| = \sqrt{1^2+1^2} = \sqrt{2}$

$\operatorname{Arg}(z) = \frac{3\pi}{4} \Rightarrow -1+i = \sqrt{2}e^{i\frac{3\pi}{4}}$.



$z = i \Rightarrow |z| = 1, \operatorname{Arg}(z) = \frac{\pi}{2} \Rightarrow i = e^{i\frac{\pi}{2}}$.

Identities for arg: ← minus!

(1) $\arg \bar{z} = -\arg z$

(2) $\arg\left(\frac{1}{z}\right) = -\arg z$

(3) $\arg(zw) = \arg z + \arg w$

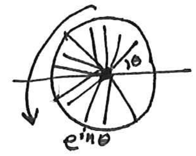
$z = re^{i\theta} \Rightarrow \bar{z} = re^{-i\theta} \Rightarrow$ OK for (1) & (2)

$w = r'e^{i\theta'}$

$zw = rr'e^{i\theta}e^{i\theta'} = rr'e^{i(\theta+\theta')} \Rightarrow$ OK.

De Moivre's formula: $\underline{(e^{i\theta})^n = e^{in\theta}}$

this implies



∴ taking n-th power ⇔ rotation

$n=2$: $(e^{i\theta})^2 = e^{2i\theta} \iff (\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$

$\iff \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta = \cos 2\theta + i\sin 2\theta$

$\iff \left.\begin{array}{l} \cos 2\theta = \cos^2\theta - \sin^2\theta \\ \sin 2\theta = 2\cos\theta\sin\theta. \end{array}\right\}$ usual fmlas for sin and cos 2θ.

$n=3$: $(e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3 \iff \dots$

$\iff \begin{array}{l} \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \\ \sin 3\theta = 3\cos^2\theta - \sin^3\theta. \end{array}$

Defn z is an n-th root of w if $z^n = w$.

$\{$n-th roots$\} = \{$solutions of $z^n - w\}$ ← there are n of them!

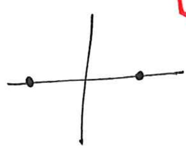
$w = \rho e^{i\varphi}$ $\quad z = r\cdot e^{i\theta}$ $\Rightarrow r^n e^{in\theta} = \rho e^{i\varphi}$

$\iff \rho = r^n$ and $\theta = \frac{\varphi}{n} + \frac{2\pi k}{n}$

$k = 0, 1, \dots, n-1.$

Ex) The square roots of 1 are 1 and -1:

$1 = e^0$, $1 = e^0$, $-1 = e^{+\pi i}$



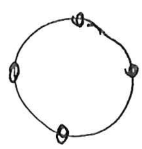
Ex) 4-th ~~square~~ roots of 1:

$e^{i\frac{\pi}{2}} = i$

$e^{2i\pi/2} = -1$

$e^{3i\pi/2} = -i$

$e^{4i\pi/2} = 1$



Ex) 4-th roots of $w = 3+3i$: $|w| = \sqrt{3^2+3^2} = \sqrt{18}$ $\quad \mathrm{Arg}(w) = \pi/4 \Rightarrow w = \sqrt{18}\, e^{i\pi/4}$

$\Rightarrow (3+3i)$ has 4 4-th roots:

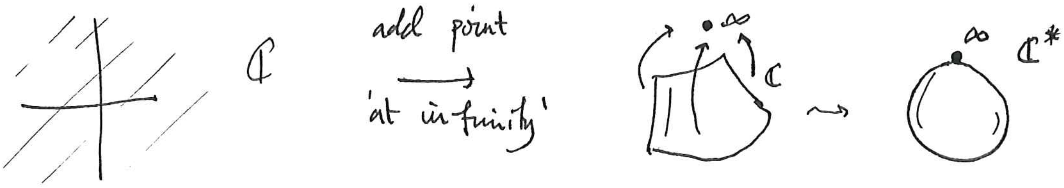
$\sqrt[8]{18}\, e^{i\pi/16}$, $-i\sqrt[8]{18}\, e^{i\pi/16}$,

$i\sqrt[8]{18}\, e^{i\pi/16}$,

$-\sqrt[8]{18}\, e^{i\pi/16}$.

Stereographic projection

Goal: Add ∞ to $\mathbb{C}$, so that $\lim_{z\to\infty}$ is well defined.

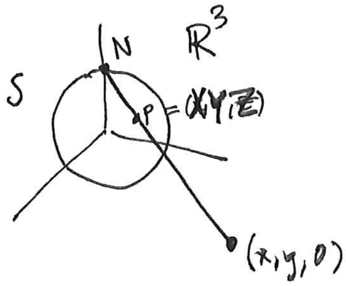


← This looks like a sphere..

"one point 'compactification'"

$\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ — the extended complex plane.

How to visualize it: Let $S \subset \mathbb{R}^3$ be the unit sphere
$X^2+Y^2+Z^2=1$



$N = (0,0,1)$

$(x,y,0)$ = intersection of the line through P and N with the $z=0$ plane.

$$N + t(P-N) = \begin{pmatrix}0\\0\\1\end{pmatrix} + t\left(\begin{pmatrix}X\\Y\\Z\end{pmatrix} - \begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \begin{pmatrix}tX\\tY\\1+t(Z-1)\end{pmatrix}$$

$$\therefore \quad \underset{\substack{\uparrow\\ z=0 \text{ plane}}}{1+t(Z-1)=0} \quad \Rightarrow \quad t = \frac{1}{1-Z}$$

$$\Rightarrow \begin{cases} x = \dfrac{X}{1-Z} \\[2mm] y = \dfrac{Y}{1-Z} \end{cases} \qquad tZ = t-1$$

We can express Z in terms of X and Y via $X^2+Y^2+Z^2=1$:

$$\overset{\cdot t^2}{\Longrightarrow} \quad (tX)^2 + (tY)^2 + (tZ)^2 = t^2 \quad \Rightarrow \quad x^2+y^2+(t-1)^2 = t^2$$

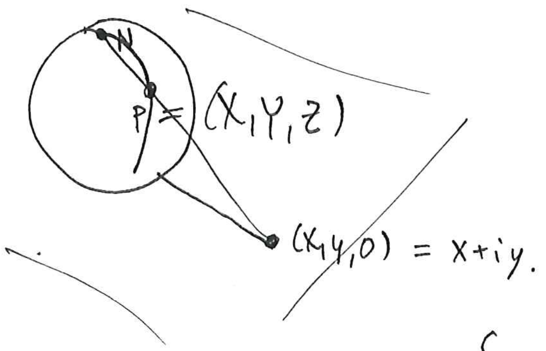
$$\Rightarrow \quad t = \frac{1}{2}\left(|z|^2+1\right)$$

$$\Rightarrow \begin{cases} X = \dfrac{2x}{|z|^2+1} \\[2mm] Y = \dfrac{2y}{|z|^2+1} \end{cases} \qquad Z = \frac{|z|^2-1}{|z|^2+1}$$

$\therefore$ 1-1 correspondence between points on S and $\mathbb{C}^*$.

Geometry:

circle on S = intersection $S \cap H$ where H is a plane



$\phi : S - N \to \mathbb{C}$

sends: circles $\ni N$ to lines in $\mathbb{C}$

circles $\not\ni N$ to circles in $\mathbb{C}$.

This is intuitive, at least geometrically

If the equation of the circle C on S is $AX + BY + CZ = D$

$$\Rightarrow \quad A \frac{2x}{|z|^2+1} + B \frac{2y}{|z|^2+1} + C \frac{|z|^2-1}{|z|^2+1} = D$$

$$\Rightarrow \quad \left(A(2x) + B(2y) + C(|z|^2-1)\right) = D(|z|^2+1) \qquad |z|^2 = x^2+y^2$$

$$\Rightarrow \quad A(2x) + B(2y) + C(x^2+y^2-1) = D(x^2+y^2+1)$$

$$\Rightarrow \quad (C-D)(x^2+y^2) + 2Ax + 2By - (C+D) = 0 \qquad (*)$$

$\therefore$ If $C = D$ then $(*)$ describes a line in $\mathbb{C}$ (this is iff $N \ni C$)

$C \neq D$ ——"—— a circle in $\mathbb{C}$ (divide by $C-D$ to see).

Conversely, given a circle in $\mathbb{C}$: $x^2+y^2+A'x+B'y+D' = 0$

Then define A, B, C, D by

$$2A = A'$$
$$2B = B'$$
$$C - D = 1$$
$$-(C+D) = D'$$

$\rightsquigarrow$ this gives a plane in $\mathbb{R}^3$ projecting onto the given circle.

Exercise 1.3.2: If P corresponds to z, then $-P$ (antipodal) corresponds to $-\frac{1}{\bar{z}}$.

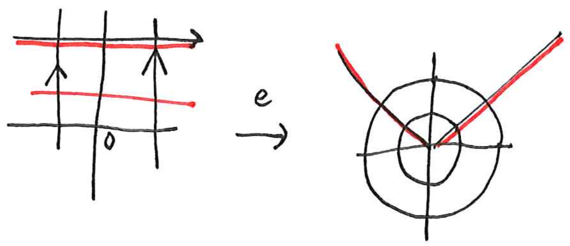
The exponential function

Defn The <u>complex exponential function</u> e^z is defined by

$$e^z = e^x(\cos y + i \sin y) \quad \text{where} \quad z = x+iy$$
$$= e^x e^{iy}$$

- $e^{z+w} = e^z \cdot e^w$
- $e^{-z} = \frac{1}{e^z}$
- $|e^z| = |e^x e^{iy}| = |e^x| = e^x$
- $e^{\bar{z}} = \overline{e^z}$

note that e^z is $2\pi i$-periodic : $e^{2\pi i + z} = e^z$.



horizontal lines $\longrightarrow$ rays

vertical lines $\longrightarrow$ circles

on the curve $y=c \Rightarrow w = e^{x+ic}$ only $|w|$ changes

on $x=c \Rightarrow w = e^{c+iy}$ only arg changes

The complex Logarithm

Defn For $z \in \mathbb{C} - 0$ the principal logarithm of z is defined by

Recall: $\operatorname{Arg} z \in (-\pi, \pi]$

$$\operatorname{Log}(z) = \log|z| + i \operatorname{Arg}(z)$$

We also write $\log z = \log|z| + i \arg z = \operatorname{Log}(z) + 2\pi i k \leftarrow$ this is <u>multivalued</u>

This is the inverse of e^z in the sense that

$$w = \log|z| + i \operatorname{Arg} z + 2\pi i k \implies e^w = e^{\log|z|}\left(e^{i \operatorname{Arg} z + i 2\pi i k}\right) = |z| e^{i \operatorname{Arg} z} = z.$$

Ex1 $z = 1+i$
$= \sqrt{2} e^{i\pi/4}$

$$\operatorname{Log} z = \underbrace{\log \sqrt{2}}_{\log|z|} + \underbrace{i \frac{\pi}{4}}_{i\operatorname{Arg}(z)}$$

$\therefore \log z = \log \sqrt{2} + \frac{\pi i}{4} + 2\pi i k \quad k \in \mathbb{Z}$

Power and phase functions

$\alpha \in \mathbb{C}$.

it is single valued if $\alpha \in \mathbb{Z}$ of course.

Defn The power function z^α is the <u>multivalued</u> function $z^\alpha = e^{\alpha \log z}$ $(z \neq 0)$

Note: $z^\alpha = e^{\alpha \log|z| + \alpha i \operatorname{Arg} z + 2\pi i k\alpha} = e^{\alpha \operatorname{Log} z} \underbrace{e^{2\pi i \alpha k}}$ $k \in \mathbb{Z}$.

get all values by multiplying with $e^{2\pi i \alpha k}$ $k = 0, \pm 1, \ldots$

Ex $3^i = e^{i \log 3} = e^{i(\log 3 + 2\pi i k)} = e^{i \log 3} e^{-2\pi k}$ $k = 0, \pm 1, \ldots$

Ex $i^i = e^{i \log i} = e^{i(\overset{=0}{\log|i|} + i \operatorname{Arg} i + 2\pi i k)} = e^{-\operatorname{Arg} i - 2\pi k} = e^{-\pi/2} \cdot e^{-2\pi k}$

these are real numbers!

Caution: these are multivalued functions, so the usual algebraic identities do not apply:

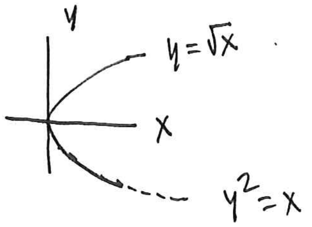
$$i^i \cdot i^{-i} = e^{2\pi n} \neq i^0 = 1.$$

$n \in \mathbb{Z}$

Geometry of $\sqrt{\ }$

even for z real, we make a choice of the solution of $w^2 = z$.

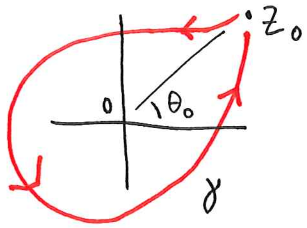
Consider the function $\sqrt{z}$. For $z \geq 0$ real, this is defined as the non-negative root.



For z complex $w = \sqrt{z}$ means $w^2 = z$; but we need to specify a choice of which square root to use.

One possibility is $w = f_1(z) = |z|^{1/2} \cdot e^{i \operatorname{Arg} z/2}$

Fix $z_0 \in \mathbb{C}$ and consider the following curve in $\mathbb{C}$: $\rightsquigarrow$ $w^2 = |z|^{2/2} \cdot e^{2i \operatorname{Arg} z/2} = |z| e^{i \operatorname{Arg} z} = z$.



As z transverses γ it is clear that θ increases from θ_0 to $\theta_0 + 2\pi$

$\rightsquigarrow$ at the end of the loop $f_1(z_0)$ has become

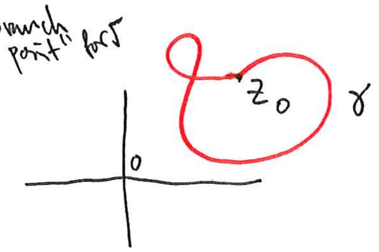
$|z|^{1/2} \cdot e^{i(\operatorname{Arg} z + 2\pi)/2} = \underbrace{|z|^{1/2} e^{i \operatorname{Arg} z}}_{z} {}^{+\pi i} = -f_1(z)$

since $f_1(z) \neq -f(z)$

$\rightsquigarrow$ our choice f_1 can not be made consistent throughout the whole $\mathbb{C}$.

The problem is $0 \in \mathbb{C}$: Consider the following curve:

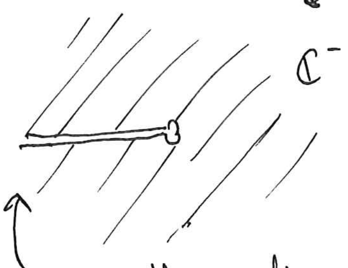
0 is a "branch point" for $\sqrt{\ }$ $\rightarrow$



This does not enclose the origin $\Rightarrow$ θ returns to its original value, θ_0, and $\sqrt{z}$ can be defined along all of the loop.

$\therefore$ To define a continuous function $\sqrt{z}$ we need to restrict to domains that do not enclose the origin. The best way to do this is to remove a curve from the origin:

Defn $\mathbb{C}^- = \mathbb{C} - [-\infty, 0]$ is the slit plane (or branch cut)



this gives a continuous function $f_1 : \mathbb{C}^- \longrightarrow \mathbb{C}$.

Note: on $\mathbb{C}^-$ we can freely define

$f_1(z) = |z|^{1/2} e^{i \operatorname{Arg} z/2}$ as the square root

When z approaches $-r$ on the negative x-axis:

$$\lim_{\varepsilon \to 0^+} f_1(-r+i\varepsilon) = i\sqrt{r}$$

$$\lim_{\varepsilon \to 0^+} f_1(-r-i\varepsilon) = -i\sqrt{r}$$

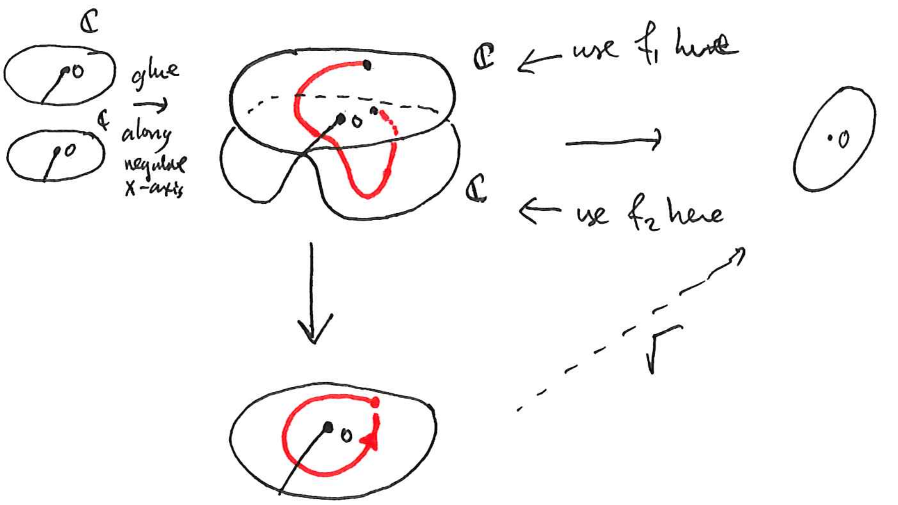
$\therefore$ the signs differ!
$\rightsquigarrow$ how to think of the extension to all of $\mathbb{C}$?

We have a second choice of the $\sqrt{\ }$, namely

$$f_2(z) = -|z|^{1/2} e^{i\operatorname{Arg} z/2} = -f_1(z)$$

Main point: $f_1(z)$ and $f_2(z)$ can be "patched" together to a well-defined single-valued function. However, it is not a function on $\mathbb{C}$, but rather a function $f: X \to \mathbb{C}$ from some other space X:

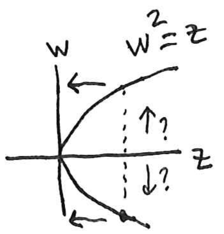
$$\begin{tikzcd}
 & X \arrow[dl, shift right] \arrow[dr, "f"] & \\
\mathbb{C} \arrow[ur, shift right, "\text{choice of branch}"'] \arrow[rr, dashed, "\sqrt{\ }"] & & \mathbb{C}
\end{tikzcd}$$



For those who have seen topology:

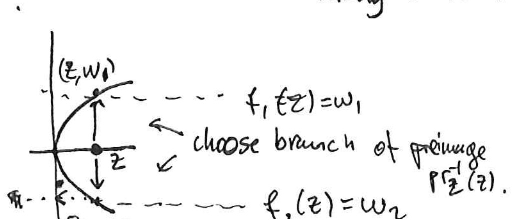
$$X = \frac{\mathbb{C} \times \{0\} \cup \mathbb{C} \times \{1\}}{(z,0) \sim (z,1) \text{ if } z \in (-\infty, 0]}$$

This is analogous to defining $X = \{(z,w) \in \mathbb{C} \times \mathbb{C} \mid w^2 = z\}$



$$\begin{tikzcd}
X \arrow[r, "pr_w"] \arrow[d, "pr_z"] & \mathbb{C} \\
\mathbb{C} \arrow[ur, dashed, "\sqrt{\ }"'] &
\end{tikzcd}$$

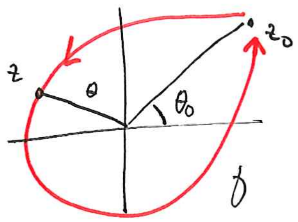
This is certainly a well-defined function $pr_w(z,w) = w$



$\therefore$ Get a well-defined "square root" $f: X \to \mathbb{C}$ given by $pr_w(z,w) = w$

Geometry of log z

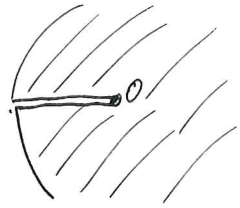
$\operatorname{Log} z = \log|z| + i\operatorname{Arg} z$ (principal logarithm)



As in the case of $\sqrt{\ }$, we see that Log is not well-defined along the whole loop γ:

θ_0 increases to $\theta_0 + 2\pi$

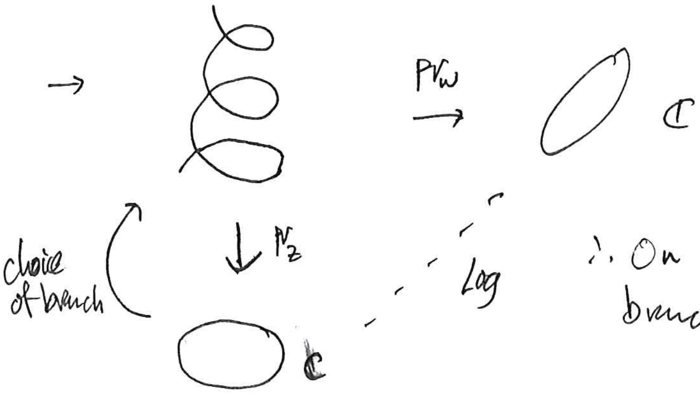
However in



← here Arg z is single valued ⇒ so is Log z.

We can make a branch cut here and form

$\{(z,w) \in \mathbb{C}^2 \mid z = \log w\}$ →



An infinite set of copies of $\mathbb{C}$ glued together along $[-\infty, 0]$. ↗

∴ On layer n, use the branch

$\log|z| + i\operatorname{Arg} z + 2\pi i n$

These constructions will not be essential in the course, but are helpful in understanding the behavior of multivalued functions.

Trigonometric and Hyperbolic functions

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$
$$\Rightarrow \quad \cos\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad (\text{for } \theta \in \mathbb{R})$$

<u>Defn</u> Define the complex cosine and sine functions by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \tan z = \frac{\sin z}{\cos z}$$

- $\cos(-z) = \cos z$ $\quad$ $\sin(-z) = -\sin z$
- $\cos(z+2\pi) = \cos(z)$ $\quad$ $\sin(z+2\pi) = \sin z$
- $\cos(z+w) = \cos z \cdot \cos w - \sin z \sin w$
  $\sin(z+w) = \sin z \cos w + \cos z \sin w$ $\quad z, w \in \mathbb{C}$
- $\sin^2 z + \cos^2 z = 1$

however $|\cos z| \leq 1$, $|\sin z| \leq 1$ are not true:

$$\cos(x+iy) = \frac{1}{2}\left(e^{ix}e^{-y} + e^{-ix}e^{y}\right) \longrightarrow \infty$$
as $y \to \pm\infty$

We also define

$$\cosh z = \frac{1}{2}\left(e^{z} - e^{-z}\right)$$
$$\sinh z = \frac{1}{2}\left(e^{z} - e^{-z}\right)$$
$$\tanh z = \frac{\sinh z}{\cosh z}$$

.. but these are not much used in this course..

The inverse sin and cos are <u>multivalued</u> functions:

$$\sin w = \frac{e^{iw} - e^{-iw}}{2} = z \quad \Rightarrow \quad e^{2iw} - 2iz e^{iw} - 1$$
$$\Rightarrow \quad e^{iw} = iz \pm \sqrt{1-z^2}$$
$$\Rightarrow \quad w = -i \log\left(iz \pm \sqrt{1-z^2}\right)$$

to understand this one must again make a branch cut.