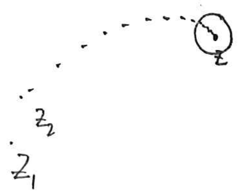
Properties of subsets in $\mathbb{C}$

$\mathbb{C}$ has a $|\ |$ (a norm) $\rightsquigarrow$ metric space $\rightsquigarrow$ can talk about limits, convergence, ...



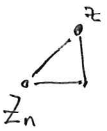
(z_n) converges to z if z_n eventually lies in any disk centred at z. $\iff$ $z_n \to z$ iff $\forall \varepsilon > 0\ \exists N > 0$ s.t $|z_n - z| < \varepsilon$ for all $n \geq N$

All the usual rules for computing limits (in $\mathbb{R}$ and $\mathbb{R}^n$) apply here (e.g $\lim_{n\to\infty} (c z_n) = c \lim z_n$)

Thm A sequence z_n converges iff $\mathrm{Re}\, z_n$ and $\mathrm{Im}\, z_n$ both converge.

proof ($\Rightarrow$):

$$|\mathrm{Re}\,(z_n - z)| \leq |z_n - z|$$
$$|\mathrm{Im}\,(z_n - z)| \leq |z_n - z|$$



triangle inequality

$\therefore$ If $z_n \to z$ then $\mathrm{Re}\, z_n \to \mathrm{Re}\, z$ and $\mathrm{Im}\, z_n \to \mathrm{Im}\, z$ also ✓

($\Leftarrow$):

$$|z_n - z| \leq |\mathrm{Im}(z_n - z)| + |\mathrm{Re}(z_n - z)|$$ (triangle inequality again)

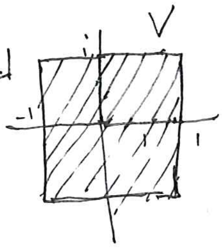
$\therefore$ If $\mathrm{Im}\, z_n \to \mathrm{Im}\, z$ and $\mathrm{Re}\, z_n \to \mathrm{Re}\, z$ then $z_n \to z$ $\square$

Defn
- $U \subseteq \mathbb{C}$ is <u>open</u> if $\forall z_0 \in U \Rightarrow \exists$ disk $B_\varepsilon(z_0)$ contained in U for some $\varepsilon > 0$
- $V \subseteq \mathbb{C}$ any set. V° = "interior of V" = $\{ z \in V \mid \exists B_\varepsilon(z) \subseteq V \}$ for some $\varepsilon > 0$

  = largest open set contained in V.
- $\overline{V}$ = "closure of V" = the set of limit points of V = $\{ z \in \mathbb{C} \mid \exists$ sequence (z_n) $z_n \in V$ s.t $z_n \to z \}$

  V is <u>closed</u> if $\overline{V} = V$.
- ∂V = "boundary of V" = $\overline{V} - V^\circ$

  = set of pts $z \in \mathbb{C}$ s.t any disk centred at z contains both points of V and $\mathbb{C} - V$.

Ex| 

$|\operatorname{Im} z| \le 1$
$|\operatorname{Re} z| \le 1$

V is closed (eg by the theorem below) $\therefore \overline{V} = V$

$V^\circ = \left\{ z \;\middle|\; \begin{array}{l} |\operatorname{Im} z| < 1 \\ |\operatorname{Re} z| < 1 \end{array} \right\}$

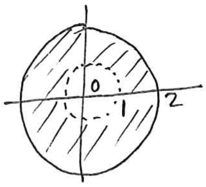
$\partial V = \{ x \pm i \mid x \in [-1,1] \} \cup \{ \cancel{x} \pm iy \mid y \in [-1,1] \}$

Ex| 

V is closed

$V^\circ = \emptyset$ (no pts in V is the center of a disk in V)

$\partial V = \overline{V} - V^\circ = V$.

Ex| 

$1 < |z| \le 2$

V neither open or closed:

$\overline{V} = \{ 1 \le |z| \le 2 \}$

$V^\circ = \{ 1 < |z| < 2 \}$ $\therefore \partial V = \{ |z| = 1 \} \cup \{ |z| = 2 \}$

strict inequalities!

Thm $V \subseteq \mathbb{C}$. V is closed $\iff$ $\mathbb{C} - V$ is open.

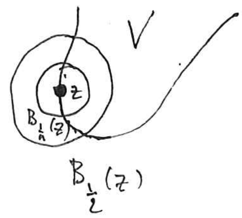
proof

($\Rightarrow$): If $\mathbb{C} - V$ is not open, there is a pt $z \in \mathbb{C} - V$ s.t any $B_\varepsilon(z)$ intersects V.

For $n \geq 1$ pick some $z_n \in B_{1/n}(z) \cap V$

$\therefore z_n \to z$ by construction, and $z_n \in V$

$\therefore z \in \overline{V} \Rightarrow V$ is not closed, since $z \in \mathbb{C} - V$.



($\Leftarrow$): If $\mathbb{C} - V$ is open, $\forall z \in \mathbb{C} - V \Rightarrow \exists B_\varepsilon(z) \subseteq \mathbb{C} - V$.

Let $z \in \overline{V}$ be a limit point, and assume that $z \notin V \Longrightarrow z \in \mathbb{C} - V$.

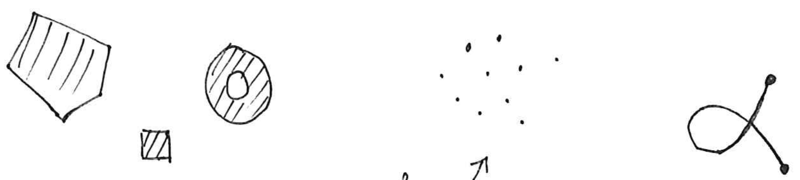
Pick $\varepsilon > 0$ such that $B_\varepsilon(z) \subseteq \mathbb{C} - V$. Then since z is a limit pt $\exists$ sequence $z_n \to z$ where $z_n \in V$. But $z_n \in B_\varepsilon(z)$ for all large n!

This contradicts the assumption that $B_\varepsilon(z) \subseteq \mathbb{C} - V$. $\square$

Defn A subset $V \subseteq \mathbb{C}$ which is both closed and bounded is said to be *compact*.

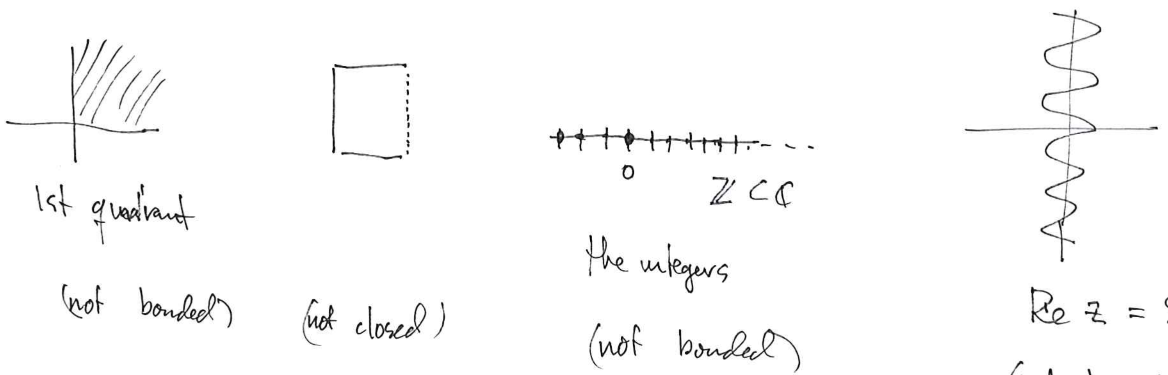
(meaning that $V \subset B_R(0)$ for some $R >> 0$)

EX 1 The following sets are compact:



finite set

EX 1 Those sets are not compact:

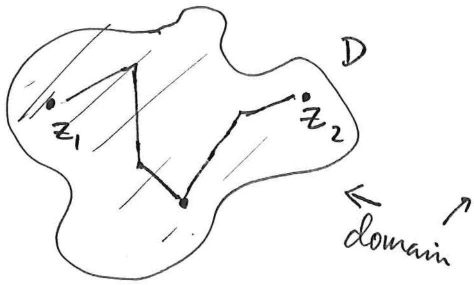


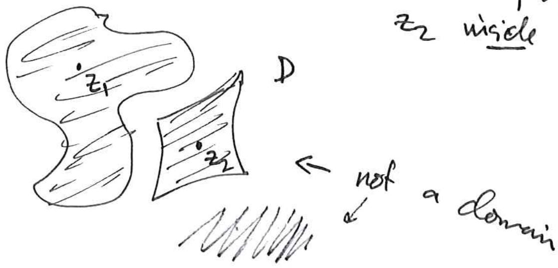
Extreme Value theorem A continuous map $f : V \to \mathbb{R}$ from a compact set V attains its maximum value at some pt in V.

EX 1 If V is not bonded then $f(z) = |z|$ is a continuous map without maximum

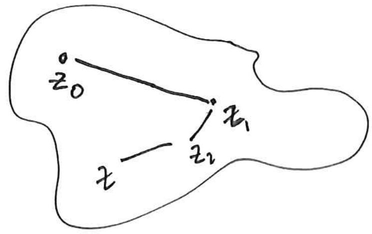
$V = \mathbb{C} - 0$ $\quad f(z) = \frac{1}{|z|}$ is unbonded (not V is not closed).

Defn $D \subseteq \mathbb{C}$ is a *domain* if D is open; and $\forall z_1, z_2 \in D \Rightarrow \exists$ broken line sequent connecting z_1, z_2 inside D



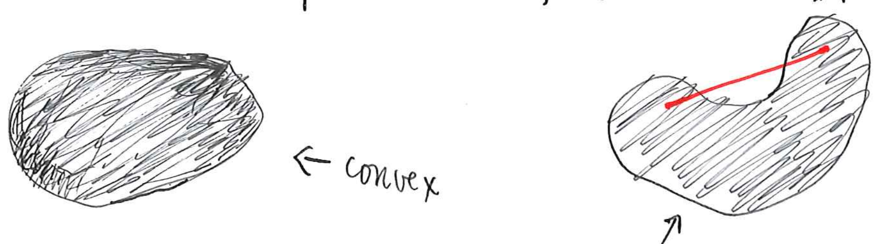


Theorem $h : D \to \mathbb{R}$ a C^1-function $h(x,y)$ in x and y, such that $\nabla h = 0$.
If D is a domain, then h is constant.

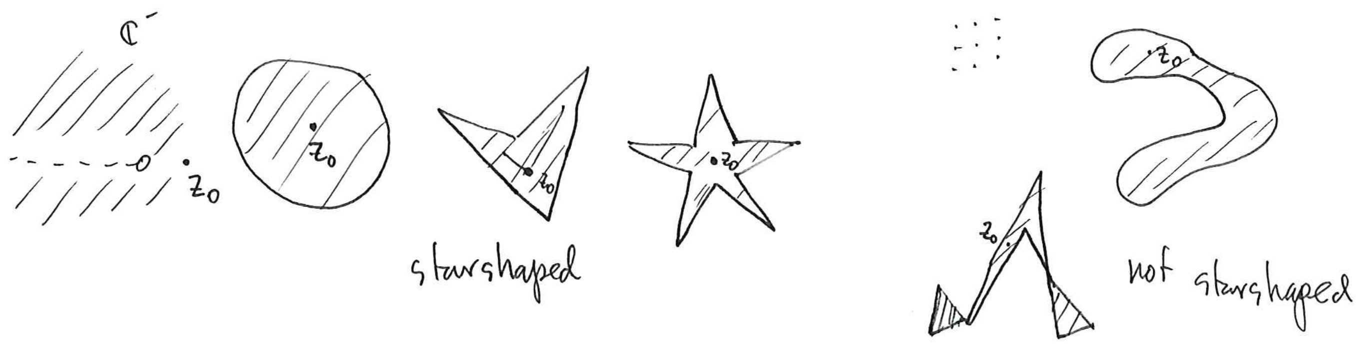


$\nabla h = 0 \Rightarrow h$ constant on any line sequent
$\Rightarrow h$ constant on all of D. $\square$

Defn D is *convex* if any two points z, w can be joined with a single straight line sequent entirely contained in D.



D is *starshaped* with respect to $z_0 \in D$ if any line sequent with z_0 as an end point (and other end point also contained in D) is contained in D:



Analytic functions

D domain in $\mathbb{C}$

<u>Defn</u> $f: D \to \mathbb{C}$ is <u>differentiable at z_0</u> if the limit

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

exists. We denote this by $f'(z_0)$ or $\frac{df}{dz}(z_0)$.

[If $f: D \to \mathbb{C}$ is differentiable at every pt in D and if $f'(z)$ is continuous on D, we say f is <u>analytic</u> in D

<u>Ex</u> $f(z) = z^n \implies \lim_{z \to z_0} \frac{z^n - z_0^n}{z - z_0} = \lim_{z \to z_0} \left(z^{n-1} + z^{n-2} z_0 + \dots + z_0^{n-1}\right)$

$$= n z_0^{n-1}. \qquad \therefore f'(z_0) = n z_0^{n-1}$$

<u>Ex</u> $f(z) = \bar{z}$ is not differentiable at any point of $\mathbb{C}$:

$$\lim_{z \to z_0} \frac{\overline{z - z_0}}{z - z_0} = \lim_{z \to 0} \frac{\bar{z}}{z}$$

If $z \to 0$ along the real axis:

$$\lim_{\varepsilon \to 0} \frac{\bar{\varepsilon}}{\varepsilon} = 1$$

If $z \to 0$ along the imaginary axis:

$$\lim_{\varepsilon \to 0} \frac{\overline{i\varepsilon}}{i\varepsilon} = -1 \implies \text{the limit does not exist.}$$

<u>Ex</u> $f(z) = |z|^2$ is differentiable at 0, but not at any other point:

$$\frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z} = \frac{(z_0 + \Delta z)\overline{(z_0 + \Delta z)} - z_0 \bar{z}_0}{\Delta z} = \bar{z}_0 + \overline{\Delta z} + z_0 \frac{\overline{\Delta z}}{\Delta z}.$$

<u>Theorem</u> $f(z)$ differentiable at $z_0 \implies f$ is continuous at z_0

← converse not true!

$$f(z) = f(z_0) + \left(\frac{f(z) - f(z_0)}{z - z_0}\right) \cdot (z - z_0)$$

Recall:

f continuous at $z_0 \iff \lim_{z \to z_0} f(z) = f(z_0)$ ← and this holds by

All the usual properties of differentiation apply also in the complex setting:

$$(cf)'(z) = c f'(z) \qquad c \in \mathbb{C}$$

$$(f+g)'(z) = f'(z) + g'(z)$$

$$(fg)'(z) = f'(z) g(z) + f(z) g'(z)$$

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z) g(z) - f(z) g'(z)}{g(z)^2} \quad \dots$$

Chain rule:

$$(f \circ g)'(z_0) = f'(g(z_0)) \cdot g'(z_0)$$

These rules are straightforward to verify, except for the Chain-rule:

proof:

Case I: $g'(z_0) \neq 0$ (hence $g(z) \neq g(z_0)$ in a small nbh of z_0)

Then we can write $$\frac{f(g(z)) - f(g(z_0))}{z - z_0} = \underbrace{\frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)}}_{\to f'(g(z_0))} \cdot \underbrace{\frac{g(z) - g(z_0)}{z - z_0}}_{\to g'(z_0)}$$

since f is differentiable at $g(z_0)$. since $g(z) \to g(z_0)$

Case II: $g'(z_0) = 0$

f differentiable at $w_0 = g(z_0)$ $\Rightarrow$ $\frac{f(w) - f(w_0)}{w - w_0}$ bounded near w_0

$$\Rightarrow \left| \frac{f(w) - f(w_0)}{w - w_0} \right| \leq C \text{ for some } C>0 \text{ and}$$

for all $0 < |w - w_0| < \varepsilon$

$$\Rightarrow |f(g(z)) - f(g(z_0))| \leq C\,|g(z) - g(z_0)|$$

$$\Rightarrow \left| \frac{f(g(z)) - f(g(z_0))}{z - z_0} \right| \leq C \cdot \left| \frac{g(z) - g(z_0)}{z - z_0} \right| \xrightarrow[z \to z_0]{} 0$$

$\Rightarrow$ $(f \circ g)'(z_0) = 0$ $\Rightarrow$ The chain rule holds.. $\square$

Ex) $F(z) = \frac{1}{z^2+3}$ Let $f(z) = \frac{1}{z}$, $g(z) = z^2+3$

$$\therefore F'(z) = (f \circ g)'(z) = f'(g(z)) \cdot g'(z) = \frac{-1}{g(z)^2} \cdot 2z = \frac{-2z}{(z^2+3)^2}$$

(same result as with the quotient rule.)

Ex) $F(z) = \left(\frac{z^2-1}{z^2+1} \right)^{100}$ $(z \neq \pm i)$

$$F'(z) = 100 \left(\frac{z^2-1}{z^2+1} \right)^{99} \cdot \frac{(z^2+1)2z - (z^2-1)2z}{(z^2+1)^2} = 400 \frac{(z^2-1)^{99}}{(z^2+1)^{101}}$$

The Cauchy Riemann equations

recall: $f'(z)$ is required to be continuous.

Let $D \subseteq \mathbb{C}$ be a domain and let $f : D \to \mathbb{C}$ be analytic. We can write

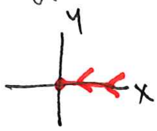
$$f(z) = u(x,y) + i\, v(x,y)$$ where u, v are continuous real functions.

Fix $z \in D$. Consider the limit:

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

How does h approach 0?

h = real : $= \Delta x$



$$\frac{f(z+\Delta x) - f(z)}{\Delta x} = \frac{u(x+\Delta x, y) + i\, v(x+\Delta x, y) - u(x,y) - i\, v(x,y)}{\Delta x}$$

$$= \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\, i$$

$$\therefore f'(z) = \frac{\partial u}{\partial x}(x,y) + i \frac{\partial v}{\partial x}(x,y) \qquad \therefore u, v \text{ are } C^1(D, \mathbb{R}).$$

h = imaginary $= i\,\Delta y$

$$\frac{f(z + i\Delta y) - f(z)}{i\,\Delta y} = \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y} - i\, \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}$$

$$\therefore f'(z) = \frac{\partial v}{\partial y}(x,y) - i \frac{\partial u}{\partial y}(x,y)$$

Hence we see that the following set of equations must be satisfied:

Dfn

The Cauchy-Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Theorem Let $f: D \to \mathbb{C}$ be a function defined on a domain $D \subseteq \mathbb{C}$.

f analytic on D $\iff$ u, v have continuous 1st order derivatives, and the Cauchy-Riemann equations hold.

proof. $\Rightarrow$ OK, from the above paragraph.

$\Leftarrow$: $f(z) = u(x,y) + i v(x,y)$.

We use Taylor's theorem for C^1-functions on D:

$$u(x+\Delta x, y+\Delta y) = u(x,y) + \frac{\partial u}{\partial x}(x,y)\cdot \Delta x + \frac{\partial u}{\partial y}(x,y)\,\Delta y + R(\Delta x, \Delta y)$$

$$v(x+\Delta x, y+\Delta y) = v(x,y) + \frac{\partial v}{\partial x}(x,y)\,\Delta x + \frac{\partial v}{\partial y}(x,y)\,\Delta y + S(\Delta x, \Delta y)$$

$\lim\limits_{\Delta z \to 0} \frac{R(\Delta x, \Delta y)}{|\Delta z|} \to 0$

$\Delta x + i\Delta y = \Delta z$

$$\therefore\ f(z+\Delta z) = f(z) + \frac{\partial u}{\partial x}(x,y)\,\Delta x + \frac{\partial u}{\partial y}(x,y)\,\Delta y + R(\Delta z)$$
$$+ i\frac{\partial v}{\partial x}(x,y)\,\Delta x + i\frac{\partial v}{\partial y}(x,y)\,\Delta y + iS(\Delta z)$$

apply CR equations here!

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$= f(z) + \left(\frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)\right)\underbrace{(\Delta x + i\Delta y)}_{=\Delta z} + R(\Delta z) + S(\Delta z)$$

$$\Rightarrow\ \frac{f(z+\Delta z) - f(z)}{\Delta z} = \left(\frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)\right) + \underbrace{\frac{R(\Delta z) + S(\Delta z)}{\Delta z}}_{\longrightarrow 0}$$

$$\longrightarrow \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)$$

$\therefore$ $f'(z)$ exists and is continuous $\Rightarrow$ f is ~~continuous~~ analytic in D $\square$

ex| $f(z) = z^2 = (x+iy)^2 = \underbrace{x^2-y^2}_{u} + \underbrace{2ixy}_{iv}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x} \qquad \therefore \text{ CR equations hold}$$

ex| $f(z) = \operatorname{Im} z = y$

$$\frac{\partial u}{\partial x} = 0 \neq \frac{\partial v}{\partial y} = 1 \qquad \therefore \text{ CR equations do not hold}$$

ex| $f(z) = \bar{z} = x - iy$

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1 \qquad \Rightarrow \bar{z} \text{ not analytic.}$$

ex| $f(z) = e^z = e^x \cos y + i e^x \sin y$

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$$

$\therefore$ f is analytic and $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$= e^x \cos y + i e^x \sin y$$

$$= e^z.$$

The Cauchy-Riemann equations give a simple criterion to determine whether or not $f(z)$ is analytic.

Some easy (but powerful) corollaries:

D domain in $\mathbb{C}$

Theorem $f: D \to \mathbb{C}$ analytic. If $f'(z) = 0$ on D then f = constant

proof. $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$ by Cauchy-Riemann equations

$\Rightarrow$ u, v constant $\Rightarrow$ $f = u + iv$ constant $\square$

$\uparrow$
D domain!

Theorem $f: D \to \mathbb{C}$ analytic and real-valued, then f = constant.

proof. $v = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$ by C-R $\Rightarrow$ f constant $\square$
since D is a domain.

Inverse mappings and the Jacobian

$f: D \longrightarrow \mathbb{C}$ analytic in D $\quad f(z) = u(x,y) + i\,v(x,y) \quad \longleftrightarrow \quad f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x,y) \longrightarrow (u(x,y), v(x,y))$

real, differentiable + CR equations

Defn The Jacobian matrix of f is $J_f = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$

Note: $\det J_f = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} \underset{\uparrow}{=} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2$

Cauchy Riemann

$= |f'(z)|^2 \geq 0$

($=0$ iff f is constant in a nbh)

Theorem f analytic $\Rightarrow \det J_f(z) = |f'(z)|^2$.

Note $f'(z_0) \neq 0 \Rightarrow |f'(z)|^2 > 0$ in a nbh of z_0.

Inverse function theorem: $f: D \to \mathbb{C}$ analytic. $z_0 \in D$ with $f'(z_0) \neq 0$.

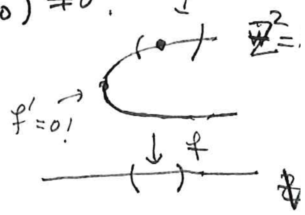
Then $\exists$ disk $U \subset D$ containing z_0 s.t $f|_U$ is 1-1 on U.

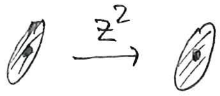
$V = f(U)$ is open and

$$g = f^{-1}: V \longrightarrow U \quad \text{is analytic}$$

with derivative

$$g'(f(z)) = (f^{-1})'(f(z)) = \frac{1}{f'(z)}.$$





(Follows from the inverse function theorem in $\mathbb{R}^n$)

$w = f(z)$
$w_1 = f(z_1)$

$$\frac{g(w) - g(w_1)}{w - w_1} = \frac{z - z_1}{f(z) - f(z_1)} = \frac{1}{\frac{f(z)-f(z_1)}{z - z_1}} \xrightarrow[z \to z_1]{} \frac{1}{f'(z_1)}$$

here $z \to z_1 \Leftrightarrow w \to w_1$ since $f|_U$ is 1-1.

$\therefore f^{-1}$ analytic in V

Ex 1 $f(z) = e^z$ on $\underset{D}{\mathbb{C}^-} = \mathbb{C} - (-\infty, 0]$.

$f'(z) = e^z \Rightarrow f'(z) \neq 0$ on all of $D \Rightarrow f$ is 1-1 around any point of D.

$g(w) = f^{-1}(w) = \operatorname{Log} w$ is a continuous inverse for f.

$$\therefore \frac{d}{dw}\operatorname{Log} w = \frac{1}{e^z} = \frac{1}{e^{\operatorname{Log} w}} = \frac{1}{w}.$$

ex1 $f(z) = \sqrt{z}$ on $D = \mathbb{C}^-$

$= e^{\frac{1}{2}\operatorname{Log} z}$

chain rule

$$\Rightarrow \frac{d}{dz} f(z) = \frac{d}{dz} e^{\frac{1}{2}\operatorname{Log} z} = e^{\frac{1}{2}\operatorname{Log} z} \frac{d}{dz}\left(\frac{1}{2}\operatorname{Log} z\right) = \underbrace{e^{\frac{1}{2}\operatorname{Log} z}}_{=\sqrt{z}} \frac{1}{2z} = \frac{1}{2\sqrt{z}}.$$

Harmonic functions

Defn The differential operator (on $\mathbb{R}^n$)

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$$

is called the Laplacian. The Laplace equation is $\Delta u = 0$.

u is harmonic if
- $u \in C^2(\mathbb{R}$
- $\Delta u = 0$

In $\mathbb{R}^2$, this becomes $\Delta u \;\; \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Thm $f(z) = u(x,y) + i\, v(x,y)$ analytic in $D \subseteq \mathbb{C}$ $\Rightarrow$ u and v are harmonic.

CR equations:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\} \Rightarrow \frac{\partial^2 u}{\partial x^2} \overset{CR}{=} \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \overset{CR}{=} -\frac{\partial^2 u}{\partial y^2} \quad \therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

If v is C^2 we are allowed to switch the order of differentiation!

Defn If u is harmonic in D and $\exists\, v$ harmonic s.t. $f = u + iv$ is analytic, we call v a harmonic conjugate of u

Note: If v is a harmonic conjugate of u $\Rightarrow$ so is $v + c$ for $c \in \mathbb{C}$.

Ex) $u = x^2 - y^2$ is harmonic: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$.

What are the harmonic conjugates of u?

$\rightsquigarrow$ Want: all $f = u + iv$ s.t. f is analytic.

f must satisfy CR: $2x = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow v = 2xy + h_1(x)$

$-2y = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow v = 2xy + h_2(y)$

$\Rightarrow v = 2xy +$ constant.

In particular, $v = 2xy$ is harmonic. What is f?

$$(x^2 - y^2) + i(2xy) = (x+iy)^2$$

$\therefore f(z) = z^2$!

D = disk or □

Theorem $u : D \to \mathbb{C}$ harmonic $\Rightarrow$ u has a harmonic conjugate.

proof. Assume for simplicity (and wlog) that $D = B_R(0)$

Define $$v = \int_0^y \frac{\partial u}{\partial x}(x,t)\,dt + \varphi(x) \qquad \left(\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \right)$$

We will determine φ s.t. $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$:

$$\frac{\partial v}{\partial x} \underset{\substack{\text{differentiating} \\ \text{under } \int \text{ sign}}}{=} \int_0^y \frac{\partial^2}{\partial x^2} u(x,t)\,dt + \varphi'(x)$$

$$\underset{u \text{ harmonic}}{=} -\int_0^y \frac{\partial^2}{\partial y^2} u(x,t)\,dt + \varphi'(x)$$

$$= -\frac{\partial u}{\partial y}(x,y) + \frac{\partial u}{\partial y}(x,0) + \varphi'(x)$$

$\therefore$ Choose φ so that $\varphi'(x) = -\frac{\partial u}{\partial y}(x,0)$.

$\therefore$ $$v(x,y) = \int_0^y \frac{\partial}{\partial x} u(x,t)\,dt - \int_0^x \frac{\partial}{\partial y}(s,0)\,ds$$ $\Leftarrow$ by construction u and v satisfy the CR equations $\Rightarrow$ v is the harmonic conjugate

□

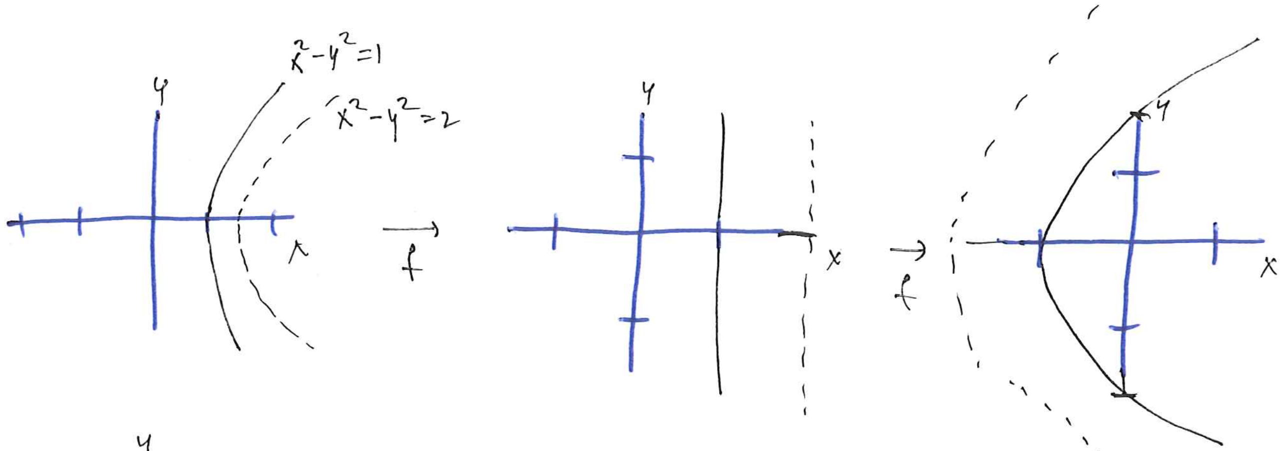
Q: Why does the proof need D = ◍ or ▨ ?

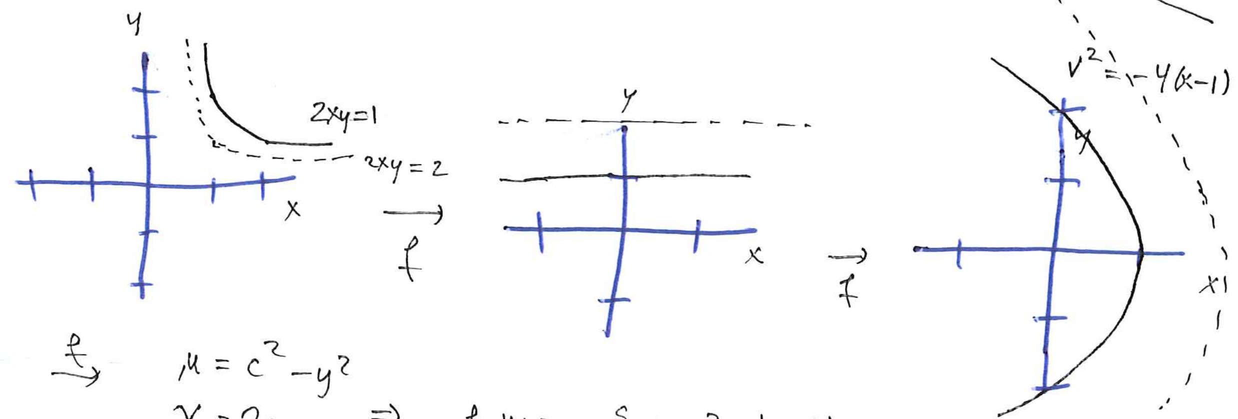
Conformal mappings [illegible]

$$f(z)=z^2 = \underbrace{(x^2-y^2)}_{u} + i\underbrace{(2xy)}_{v}$$

$\therefore$ f maps the hyperbolas $x^2-y^2=c$, $2xy=d$ to $u=c$, $v=d$ respectively

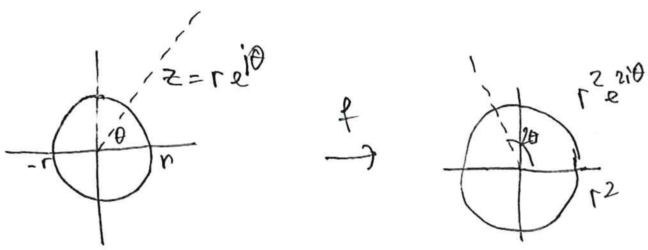
Interesting fact: for $c,d\neq 0$ these hyperbolas intersect at right angles!





$x=c \xrightarrow{f}$ $u=c^2-y^2$, $v=2cy$ $\Rightarrow$ f maps $\{x=c\}$ to the parabola $v^2=-4c^2(u-c^2)$

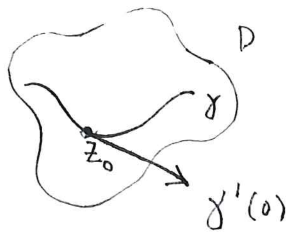
$y=d \xrightarrow{f}$ $v^2=4d^2(u+d^2)$ $\leftarrow$ parabolas in w-plane intersect at $(c^2-d^2, \pm 2|cd|)$



$\therefore$ f maps $\{|z|=r\}$ to $\{|w|=r^2\}$

$\therefore$

Conformal mappings



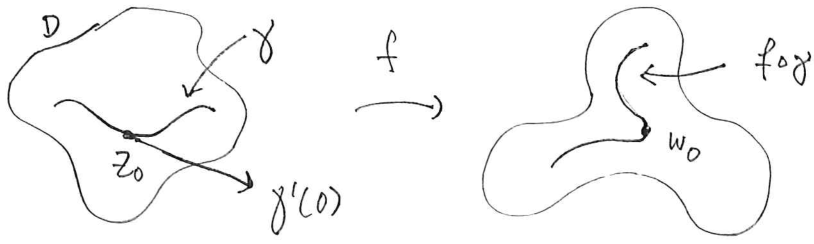
$\gamma : [-1,1] \longrightarrow D$ smooth curve in D

$z_0 = \gamma(0)$

$\gamma(t) = x(t) + i\,y(t)$ $\qquad$ x, y are C^∞.

$$\Rightarrow \quad \gamma'(0) = \lim_{t \to 0} \frac{\gamma(t) - \gamma(0)}{t} = x'(t) + i\,y'(t)$$

$f : D \longrightarrow \mathbb{C}$ analytic



z $\qquad\qquad$ $w = f(z)$

$\rightsquigarrow \quad f \circ \gamma : [-1,1] \longrightarrow \mathbb{C}$ curve in the complex plane.

What is the tangent vector of $(f \circ \gamma)$ at w_0?

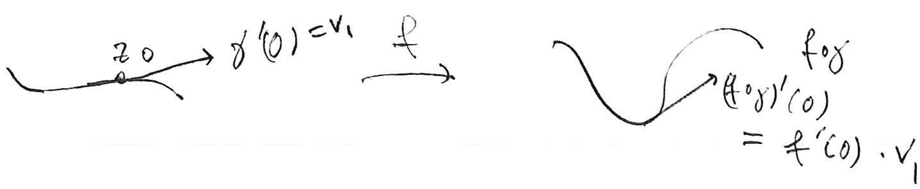
Chain rule: $(f \circ \gamma)'(0) = \frac{df}{dz}(\gamma(t)) \cdot \frac{d\gamma}{dt} = f'(z_0) \cdot \gamma'(0)$

$\therefore$ Tangent vectors get multiplied by $f'(z_0)$

Write $\quad f'(z_0) = r e^{i\theta}$

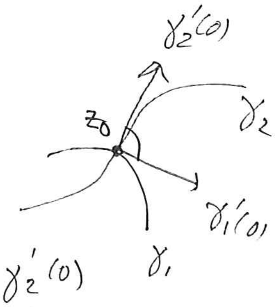
$\qquad\quad \gamma'(0) = v \qquad \Rightarrow$ new tangent vector is $r e^{i\theta} \cdot v$.

$\Longrightarrow$ new tangent vector is stretched by a factor of $r = |(f \circ \gamma)'(0)|$ and rotated by an angle of $\theta = \operatorname{Arg}((f \circ \gamma)'(0))$



What happens to angles under analytic maps $f: D \to \mathbb{C}$?

$\gamma_1 : (-1,1) \to \mathbb{C}$
$\gamma_2 : (-1,1) \to \mathbb{C}$
smooth paths s.t. $\gamma_1(0) = \gamma_2(0)$, $\gamma_1'(0) \neq 0$, $\gamma_2'(0) \neq 0$



Defn The (directed) angle between the paths γ_1 and γ_2 at z_0 is defined as

$$\arg \gamma_2'(0) - \arg \gamma_1'(0)$$

Alternatively, one can define this as θ s.t. $e^{i\theta} = \dfrac{w/|w|}{z/|z|}$ $\quad w = \gamma_2'(0)$, $z = \gamma_1'(0)$.

$$\begin{tikzcd} (-1,1) \arrow[r, "\gamma"] \arrow[rr, bend right, "\sigma"'] & \mathbb{C} \arrow[r, "f"] & \mathbb{C} \end{tikzcd}$$

$\sigma'(t) = f'(\gamma(t)) \cdot \gamma'(t)$ $\qquad \arg \sigma'(0) = \arg f'(z_0) + \arg \gamma'(0)$

$$\therefore \quad \arg \sigma'(0) - \arg \gamma'(0) = \arg f'(z_0) \qquad (*)$$

$\gamma_1, \gamma_2 \rightsquigarrow$ two paths σ_1, σ_2

$$\arg \sigma_1'(0) - \arg \gamma_1'(0) = \arg f'(z_0)$$
$$\arg \sigma_2'(0) - \arg \gamma_2'(0) = \arg f'(z_0)$$

$\longrightarrow \quad \arg \sigma_1'(0) - \arg \sigma_2'(0) = \arg \gamma_1'(0) - \arg \gamma_2'(0)$

$\rightsquigarrow$ the directed angles are the same!

Prop If $f: D \to \mathbb{C}$ is analytic, then f preserves angles at each point of D where $f'(z) \neq 0$.

$f: D \to V$ $\mathbb{C}^1$ (continuously differentiable) is conformal if preserves angles and is 1-1 onto V

$\therefore$ analytic functions are (locally) conformal at points where $f'(z) \neq 0$

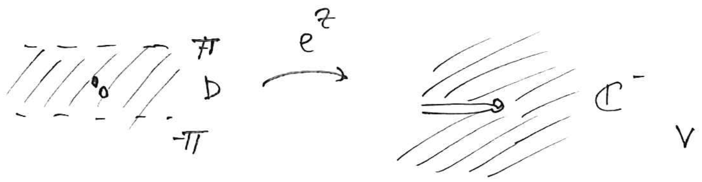
ex) $f(z) = az$
$f(z) = z+b$ are conformal everywhere (clearly preserve angles)

ex) $f(z) = \bar{z}$ reverses angles $\Rightarrow$ not conformal!

ex) $f(z) = z^n$ not conformal at $z=0$ (but everywhere else).

ex) $f(z) = e^z$ not conformal mapping since not 1-1 onto $\mathbb{C}-0$
$f'(z) = e^z \neq 0$.

$f\big|_D$ is conformal

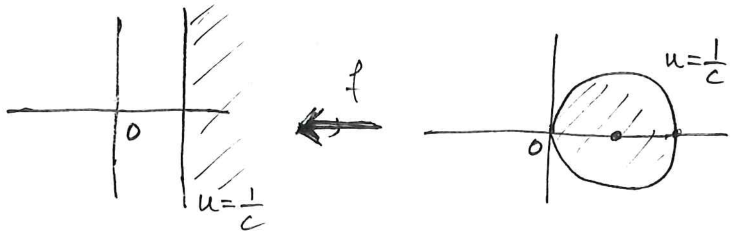


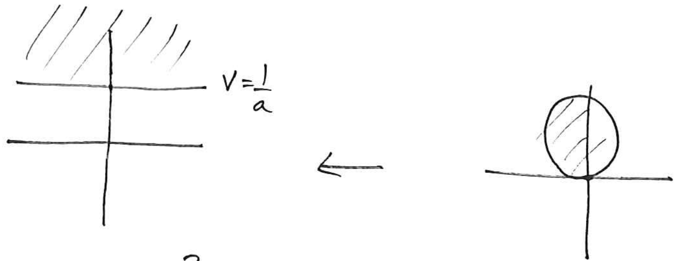
Ex] $f(z) = \frac{1}{z} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}\, i$ conformal at $D = \mathbb{C} - 0$:

- conformal at every pt (and
- 1-1 onto $\mathbb{C} - 0$.

"level sets": $u = \text{const} = \frac{1}{c}$

$$\frac{x}{x^2+y^2} = \frac{1}{c} \Rightarrow \frac{1}{c}(x^2+y^2) = x \Rightarrow \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$$





$$\frac{-y}{x^2+y^2} = \frac{1}{a} \Rightarrow x^2 + \left(y + \frac{a}{2}\right)^2 = \frac{a^2}{4}$$

$\therefore$ circle at $(0, -\frac{a}{2})$ radius $|\frac{a}{2}|$

Möbius transformations

Defn A function of the form

$$f(z) = \frac{az+b}{cz+d} \qquad a,b,c,d \in \mathbb{C}, \quad ad-bc \neq 0$$

$\leftarrow$ =: the determinant of f.

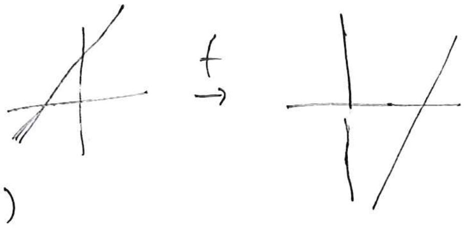
note (ka, kb, kc, kd) defines the same f as (a,b,c,d) $\quad k \in \mathbb{C}-0$.

is called a Möbius transformation (or Fractional linear transformation (FLT))

$\leftarrow$ for FLTs the condition $ad-bc \neq 0$ is usually not required

$$f'(z) = \frac{ad-bc}{(cz+d)^2} \longrightarrow f(z) \text{ not constant.}$$

Ex $f(z) = az+b$ $\quad a \neq 0$ $\quad$ affine transformation:



$f(z) = z+b$ $\quad$ translation

$f(z) = az$ $\quad$ dilation $\quad$ (when $a = e^{i\theta}$ f is a rotation!)

$f(z) = \frac{1}{z}$ $\quad$ inversion $\leftarrow$ we will study this later.

$f(z)$ is not defined at points where $cz+d$: Not a function $\mathbb{C} \to \mathbb{C}$

But we can define a function

$$f : \mathbb{C}^* \longrightarrow \mathbb{C}^* \qquad \mathbb{C}^* = \mathbb{C} \cup \{\infty\}$$

extended complex plane.

If f is affine ($\Leftrightarrow c=0$): $\quad f(\infty) = \infty$

If not: $c \neq 0$

$$f(z) = \begin{cases} \frac{az+b}{cz+d} & z \neq -\frac{d}{c}, \infty \\ \infty & z = -d/c \\ \frac{a}{c} & z = \infty \end{cases}$$

$\lim \frac{az+b}{cz+d} = \frac{a}{c}$

$\therefore$ Inversion $z \to \frac{1}{z}$ interchanges 0 and ∞.

prop $f, g(z)$ two Möbius transformations

(1) $f^{-1}(z)$ is a Möbius transformation (as a map $\mathbb{C}^* \to \mathbb{C}^*$)

(2) $f \circ g$ is a Möbius transformation

proof: (1) $\quad w = \frac{az+b}{cz+d} \rightsquigarrow z = \frac{-dw+b}{cw-a} = f^{-1}(w)$

non-zero determinant: $(-d)(-a) - bc = ad-bc \neq 0$.

$\therefore$ Möbius transformations are 1-1 as functions $\mathbb{C}^* \longrightarrow \mathbb{C}^*$

(2) $f(z) = \frac{az+b}{cz+d}$ $\quad g(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$

$$f(g(z)) = \frac{a\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) + b}{c\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) + \delta} = \frac{(a\alpha + b\gamma)z + a\beta + b\delta}{(c\alpha + d\gamma)z + c\beta + d\delta}.$$

$\therefore$ If $f \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $g \leftrightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then

$$(f \circ g) \Longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$\therefore$ If $\det f \neq 0$ and $\det g \neq 0$ then $\det(f \circ g) \neq 0$ $\square$

The set of Möbius transformations forms a group (has identity: z, multiplication = composition, inverses exist).

$f : \mathbb{C}^* \to \mathbb{C}^*$ Möbius transformation

What are the fixpoints of f? $f(z) = z$

$$\frac{az+b}{cz+d} = z \implies cz^2 + (d-a)z - b = 0$$

$\therefore f$ has ≤ 2 fixed pts (unless $f(z) = z$)

$\therefore$ if f is a FLT with ≥ 3 fixed pts then $f(z) = z$.

Let now $a, b, c \in \mathbb{C}^*$ distinct.

$$\alpha = f(a)$$
$$\beta = f(b)$$
$$\gamma = f(c)$$

If g is another Möbius transformation s.t. $\alpha = g(a)$, $\beta = g(b)$, $\gamma = f(c)$

$\implies g^{-1} \circ f$ has a, b, c as fixed pts $\implies g^{-1} \circ f = \mathrm{Id} \implies g = f$.

$\therefore$ A Möbius transformation is uniquely determined by its action on any 3 pts in $\mathbb{C}^*$.

Let $z_2, z_3, z_4 \in \mathbb{C}^*$. Define $f : \mathbb{C}^* \to \mathbb{C}^*$ by

$$f(z) = \left(\frac{z - z_3}{z - z_4}\right) \Big/ \left(\frac{z_2 - z_3}{z_2 - z_4}\right) \quad \text{if } z_2, z_3, z_4 \in \mathbb{C}$$

$$f(z) = \frac{z - z_3}{z - z_4} \quad \text{if } z_2 = \infty$$

$$f(z) = \frac{z_2 - z_4}{z - z_4} \quad \text{if } z_3 = \infty \qquad (*)$$

$$f(z) = \frac{z - z_3}{z_2 - z_3} \quad \text{if } z_4 = \infty$$

these follows from the top formula by taking limits as $z_2 \to \infty$, $z_3 \to \infty$, $z_4 \to \infty$ respectively..

We have
$$f(z_2) = 1$$
$$f(z_3) = 0$$
$$f(z_4) = \infty$$

Defn $z_1 \in \mathbb{C}^*$.

We define the cross-ratio, denoted by (z_1, z_2, z_3, z_4), as $f(z_1)$ where

f = unique Möbius transformation s.t $f(z_2) = 1$, $f(z_3) = 0$, $f(z_4) = \infty$. (f is defined in (*))

Ex
- $(z_2, z_2, z_3, z_4) = 1$ (by def)
- $(z, 1, 0, \infty) = z$ ($f(z) = z$ is the FLT that ..)
- $(z, z_1, z_2, z_3) = f(z)$ is a Möbius transformation satisfies $f(z_1) = 1$, $f(z_2) = 0$, $f(z_3) = \infty$

Prop $z_2, z_3, z_4 \in \mathbb{C}^*$ distinct pts, $f(z)$ a Möbius transformation, then $(z_1, z_2, z_3, z_4) = (f(z_1), f(z_2), f(z_3), f(z_4))$

proof. Let $g(z) = (z, z_2, z_3, z_4)$ $\therefore$ g is a Möbius trnsf.

$h = g \circ f^{-1}$ $\Rightarrow$ $h(f(z_2)) = 1$, $h(f(z_3)) = 0$, $h(f(z_4)) = \infty$ $\Rightarrow$ $h(z) = (z, f(z_2), f(z_3), f(z_4))$ $\forall z \in \mathbb{C}^*$

Now substitute $z = f(z_1)$.. $\square$

Thm $z_2, z_3, z_4 \in \mathbb{C}^*$ distinct

$w_2, w_3, w_4 \in \mathbb{C}^*$ distinct

$\Rightarrow$ $\exists$ unique Möbius transformation f s.t $f(z_2) = w_2$, $f(z_3) = w_3$, $f(z_4) = w_4$

proof. Let $g(z) = (z, z_2, z_3, z_4)$

$h(z) = (z, w_2, w_3, w_4)$

$f(z) = (h^{-1} \circ g)(z)$ $\leftarrow$ f has the desired property.

Uniqueness: If $r(z)$ is another such Möbius transformation with, then $r^{-1} \circ f$ has 3 fixed pts $\Rightarrow$ $r = f$. $\square$

EX) Find $f : \mathbb{C}^* \to \mathbb{C}^*$ (Möbius) s.t

$$f(1) = 0$$
$$f(2) = i$$
$$f(3) = \infty$$

$\frac{1}{z-3}$ sends 3 to ∞

$z-1$ sends 1 to 0 $\longrightarrow$ try $f(z) = c \cdot \frac{z-1}{z-3}$

$$f(2) = i = c\frac{2-1}{2-3} = c\cdot(-1) \quad \therefore c = -i$$

$\therefore$ $f(z) = -i\,\frac{z-1}{z-3}$ works.

EX) Here's a nice trick to find f given (z_2, z_3, z_4) and (w_2, w_3, w_4)

Use the fact that f preserves the cross-ratio:

$$(w, w_2, w_3, w_4) = (f(z), f(z_2), f(z_3), f(z_4)) = (z, z_2, z_3, z_4)$$

$\rightsquigarrow$ solve for w!

$(z_2, z_3, z_4) = (i, \infty, 3)$

$(w_2, w_3, w_4) = (\infty, 0, 1)$ $\Rightarrow$ $\frac{1-0}{1-w} = \frac{i-z}{3-z} = \frac{z-i}{z-3}$

$\Rightarrow$ $\frac{z-3}{z-i} = 1-w$ $\therefore$ $w = 1 - \frac{z-3}{z-i}$

$$= \frac{(z-i)-(z-3)}{z-i}$$

$$= \frac{3-i}{z-i} \text{ does the job.}$$

Ex) $(z_2, z_3, z_4) = (0, i, \infty)$

$(w_2, w_3, w_4) = (-1, 0, 1)$

$$\left(\frac{w-0}{w-1}\right) \Big/ \left(\frac{-1-0}{-1-1}\right) = 2\frac{w}{w-1} = \frac{z-i}{0-i} = i(z-i)$$

$\Rightarrow$ $2w = (iz+1)(w-1) \Rightarrow (2-(iz+1))w = -iz-1$

$$\Rightarrow \quad w = \frac{-iz-1}{2-iz-1} = \frac{-1-iz}{1-iz} \overset{\cdot i}{=} \frac{z-i}{i+z}$$

Thm Any Möbius transformation is a composition of

- $z \to cz$ (dilation)
- $z \to z+b$ (translation)
- $z \to \frac{1}{z}$ (inversion)

Proof. CASE I $c=0$: $f(z) = \frac{az+b}{d} = \frac{a}{d}z + \frac{b}{d}$ composition:

$$z \to \frac{a}{d}z \to \frac{a}{d}z + \frac{b}{d}$$

CASE II: $c \neq 0$ Can assume $c=1$ (scaling).

$$\frac{az+b}{z+d} = a + \frac{b-ad}{z+d}$$

$$z \to z+d \to \frac{1}{z+d} \to \frac{b-ad}{z+d} \to \frac{b-ad}{z+d} + a \quad \square$$

Cor $f: \mathbb{C}^* \to \mathbb{C}^*$ maps circles in $\mathbb{C}^*$ to circles

Here "circle" in $\mathbb{C}^*$ includes straight lines.

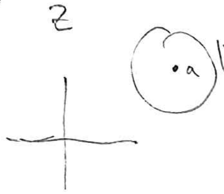
proof suffices to check it for

$z \to cz$ ✓

$z \to z+b$ ✓

$z \to \frac{1}{z}$

Case I



$|z-a|^2 = r^2$

$\rightsquigarrow$ $f(z) = \frac{1}{z} = w$

$$|w^{-1}-a|^2 = r^2 \iff |wa-1|^2 = r^2|w|^2$$

$$\iff 0 = (wa-1)(\bar{w}\bar{a}-1) - r^2|w|^2$$

$$\iff (|a|^2-r^2)|w|^2 - aw - \bar{a}\bar{w} + 1 = 0$$

$w = u+iv$

$$\iff (|a|^2-r^2)(u^2+v^2) - Au - Bv + 1$$

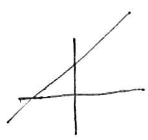
$A = a + \bar{a} \in \mathbb{R}$

$B = ai - i\bar{a} = \overline{ai} + ai \in \mathbb{R}$

$r = |a| \Rightarrow$ image = straight line in $\mathbb{C}$

$r \neq |a| \Rightarrow$ image = circle in $\mathbb{C}$

Case II



$Cx + Dy = E$

$$\rightsquigarrow C\,\mathrm{Re}\left(\frac{z+\bar{z}}{2}\right) + D\,\mathrm{Im}\left(\frac{z-\bar{z}}{2i}\right) = E$$

$$\rightsquigarrow C\,\mathrm{Re}\left(\frac{w^{-1}+\bar{w}^{-1}}{2}\right) + D\,\mathrm{Im}\left(\frac{w^{-1}-\bar{w}^{-1}}{2i}\right) = E$$

Write $w = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{(-y)}{x^2+y^2} = a + ib$

$\Rightarrow \quad x = \frac{a}{a^2+b^2} \qquad y = -\frac{b}{a^2+b^2}$ (by symmetry of z and w)

$C\frac{a}{a^2+b^2} - D\frac{b}{a^2+b^2} = E \quad \Rightarrow \quad Ca - Db = E(a^2+b^2)$

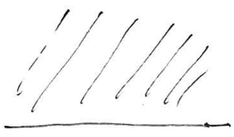
equation of a circle if $E \neq 0$

— " — a line if $E = 0$

$\square$

Ex) $f(z) = \dfrac{z-i}{z+i}$ $\quad \mathbb{H} = \{ z \mid \mathrm{Im}\, z > 0 \}$ ← open set in $\mathbb{C}$

∴ the upper half plane

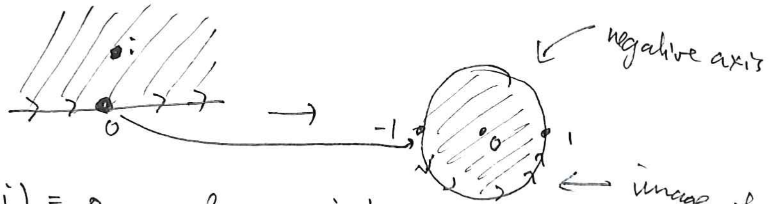


f(real axis):

$z = x \quad f(z) = \dfrac{x-i}{x+i} = \dfrac{(x-i)(x-i)}{x^2+1} = \dfrac{(x^2-1)}{x^2+1} + \dfrac{2i}{x^2+1}$

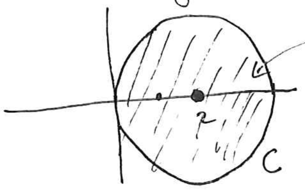
$\left(\dfrac{t^2-1}{t^2+1}, \dfrac{2t}{t^2+t} \right)$ parametrizes the unit circle $\Rightarrow$ f(real axis) $= \partial B_1(0)$

$=$ unit circle.



$f(i) = 0$ + f maps interior to interior. ← image of positive axis

Ex1 Find the image of D under $w = f(z) = \dfrac{z}{2z-8}$



$D = \{ |z-2| < 2 \}$

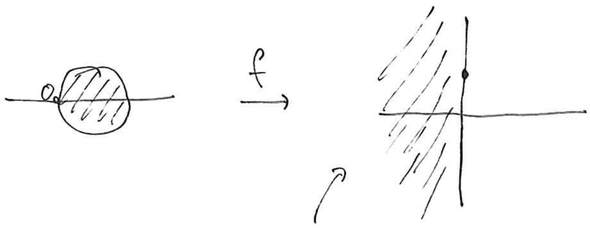
Finding $f(C)$: $\quad f(z)$ has a pole at $z=4$

$\lim_{z \to 4} f(z) = \infty \quad \therefore$ Image of C is a straight line

Which line? $\quad z = 0: \quad f(0) = 0$

$\quad z = 2+2i \quad f(2+2i) = \dfrac{2+2i}{2(2+2i)-8} = -\dfrac{i}{2}$

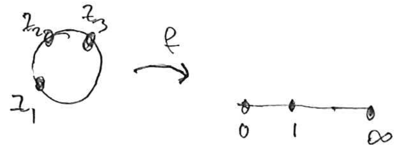
$\therefore$



$f(D)$ is the left-hand plane.

Geometry from Möbius transformations

Lemma Given a circle or line C in $\mathbb{C}$ $\rightarrow$ $\exists!\, f : \mathbb{C}^* \to \mathbb{C}^*$ Möbius s.t. $f(C) = \{\operatorname{Im} z = 0\}$



- f transitive on triples of pts
- f preserves circles
- there is a unique circle through 3 pts.

Lemma Given z_1, z_2, z_3, z_4 then $(z_1, z_2, z_3, z_4) \in \mathbb{R}$ iff $z_1 \dots z_4$ lie on a circle

For $z_1 z_2 z_3$ let C be the circle containing them

Using $f : \mathbb{C}^* \to \mathbb{C}^*$ we may assume $(z_1 z_2 z_3) = (0, 1, \infty)$

$$\Rightarrow \quad (z_1 z_2 z_3 z_4) = (0\ 1\ \infty\ z_4) = \frac{1 - z_4}{0 - z_4} = -\frac{1}{z_4} + 1$$

This is in $\mathbb{R}$ iff $z_4 \in \mathbb{R}$.

Now, we have for any $z_1, z_2, z_3, z_4 \in \mathbb{C}$

$$(z_1 - z_2)(z_3 - z_4) + (z_1 - z_4)(z_2 - z_3) = (z_1 - z_3)(z_2 - z_4)$$

Apply the triangle inequality:

$$|z_1 - z_2||z_3 - z_4| + |z_1 - z_4||z_2 - z_3| \geq |z_1 - z_3||z_2 - z_4| \quad (*)$$

When do we have equality? Need equality in $|z| + |w| \geq |z + w| \Leftrightarrow \frac{z}{w} \in \mathbb{R}_{>0}$

$\therefore$ equality in $(*)$ $\Leftrightarrow$ $\dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \in \mathbb{R}_{>0}$

$\Leftrightarrow$ $(z_1, z_3, z_2, z_4) \in \mathbb{R}_{<0}$

$\Leftrightarrow$ $z_1\ z_2\ z_3\ z_4$ lie on a circle and z_1 is opposite z_3

. Ptolemy's inequality: For A, B, C, D points in the plane

$$\overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{DA} \geq \overline{AC} \cdot \overline{BD}$$ with $=$ iff A, B, C, D are cocyclic

Problem 4 on the Exam for MAT2410 H15:

"Let $\mathbb{C}^*$ be the extended complex plane. Consider the Möbius transformation

$f : \mathbb{C}^* \to \mathbb{C}^*$ given by

$$f(z) = \frac{z-1}{z+1}$$

a) Find the fixed pts of f and $f(\infty)$

b) Find the Möbius transformation with the property

$g(-i) = -i$
$g(i) = i$
$g(1) = \infty$ "

Solution:
a)
- $f(\infty) = \lim_{z \to \infty} \frac{z-1}{z+1} = 1$
- Fixed pts: $f(z) = \frac{z-1}{z+1} = z \quad \therefore \quad z - 1 = z^2 + z$

$\therefore \quad z^2 = -1 \quad \therefore \quad \underline{z = \pm i}$

b) Note that $f(i) = i$, $f(-i) = -i$ from a)
$f(1) = 0$

Try $g(z) = \frac{-1}{f(z)}$: $\quad g(i) = \frac{-1}{i} = i$

$g(-i) = \frac{-1}{-i} = -i$

$g(1) = \lim_{z \to 1} -\frac{z+1}{z-1} = \infty \quad$ ✓

Alternatively, use cross ratios:

$(z, i, -i, 1) = (w, -i, i, \infty)$

$$\left(\frac{z+i}{z-1}\right) \Big/ \underbrace{\frac{i-(-i)}{i-1}}_{\frac{2i}{i-1}} = \frac{w-(-i)}{\cancel{w} - (-i)} = \frac{w+i}{\cancel{w}+i} = \frac{w+i}{2i}$$

$$\therefore \quad w + i = \cancel{2i} \cdot \frac{i-1}{\cancel{2i}} \frac{z+i}{z-1} = \frac{(i-1)(z+i)}{z-1}$$

$$\therefore \quad w = \frac{(i-1)(z+i)}{z-1} - i = \frac{(i-1)(z+i) - i(z-1)}{z-1}$$

$$= \frac{z(\cancel{i}-1-\cancel{i}) + i(i-1) + i}{z-1} = \frac{-z-1}{z-1} = -\frac{z+1}{z-1}$$