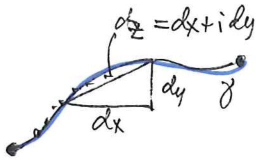
Chapter IV (Recap):

$\gamma : I \to \mathbb{C}$ smooth curve, $I = (a,b)$ interval



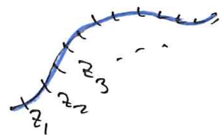
Integration in $\mathbb{R}^2$ ⇝ has Green's theorem..

$$\int_\gamma h(z)\,dz := \int_\gamma h(z)\,dx + i \int_\gamma h(z)\,dy$$

F: Riemann-sum approximation $\int_\gamma h(z)\,dz \approx \sum h(z_j)(z_{j+1} - z_j)$

In particular, we have the length of γ: $|dz| = \sqrt{(dx)^2 + (dy)^2}$

$$L = \int_\gamma 1|dz| = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$



ML estimate:

$$\left| \int_\gamma h(z)\,dz \right| \le \int_\gamma |h(z)|\,|dz| \le ML$$

"triangle inequality for integrals".

$M = \sup_{z \in \gamma} h(z)$

$L =$ length of γ.

Fundamental theorem of calculus for analytic functions:

F is a complex primitive for $f(z)$ if $F(z)$ is analytic and $F'(z) = f(z)$.

Then

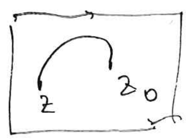
$$\int_\gamma f(z)\,dz = F(B) - F(A)$$

γ any path from A to B.

Such F's always exist for nice domains:

D star shaped:

$$F(z) := \int_{z_0}^{z} f(w)\,dw$$



take the integral along any path from z_0 to z. D starshaped implies that this is well-defined.

[Recall: $P\,dx + Q\,dy$ closed $\iff \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$]

Cauchy's theorem: $f(z) = u + iv$

$$f(z)\,dz = (u+iv)(dx+i\,dy) = (u+iv)\,dx + (-v+iu)\,dy$$

$f(z)\,dz$ closed differential $\iff \dfrac{\partial}{\partial y}(u+iv) = \dfrac{\partial}{\partial x}(-v+iu) \iff$ CR eqs.

∴ Thm f is analytic (differentiable on D) $\iff$ $f(z)\,dz$ is closed.

Green's theorem $\longrightarrow$

[$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$]

needed to integrate along the boundary

Thm (Cauchy's theorem) D bounded domain with smooth ∂D. $f(z)$ analytic on D with a smooth extension to ∂D

$$\Rightarrow \int_{\partial D} f(z)\,dz = 0.$$

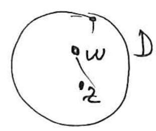
This in turn implies the following fundamental theorem:

Thm (Cauchy integral formula) D bounded domain

∂D smooth

f analytic on D

$f|_{\partial D}$ smooth extension of f.



$$\Rightarrow \boxed{f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\,dw \qquad \forall z \in D}$$

← The most important formula in the whole course!

Idea of proof:

$D_\varepsilon = D - \{|w-z| \le \varepsilon\}$

$\partial D_\varepsilon = \partial D \cup \{|z-w| = \varepsilon\}$

and $\dfrac{f(w)}{w-z}$ is analytic for all $w \in D_\varepsilon$.

$$\therefore \int_{\partial D_\varepsilon} \frac{f(w)}{w-z}\,dz = 0 \text{ by Cauchy.}$$

$$\therefore \int_{|w-z|=\varepsilon} \frac{f(w)}{w-z}\,dw = \int_{\partial D} \frac{f(w)}{w-z}\,dw \qquad \begin{aligned} w &= z + \varepsilon e^{i\theta} \\ dw &= i\varepsilon e^{i\theta}\,d\theta \end{aligned}$$

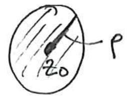
$$\Rightarrow \underbrace{\int_0^{2\pi} f(z+\varepsilon e^{i\theta})\frac{d\theta}{2\pi}}_{\text{mean value of } f(w) \text{ on } D(z) \xrightarrow{\varepsilon\to 0} f(z)} = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\,dw \xRightarrow{\text{take } \varepsilon\to 0} f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\,dw \qquad \square$$

We can differentiate under $\int$ and use $\dfrac{d^m}{dz^m}\left(\dfrac{1}{w-z}\right) = \dfrac{m!}{(w-z)^{m+1}}$

$$\rightsquigarrow \quad f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}\, dw$$

$\therefore$ f analytic $\Rightarrow$ f is infinitely differentiable and all $f', f^{(2)}, \dots$ are also analytic.

Liouville's theorem and the Cauchy estimates



$\overline{D}_\rho(z_0) = \{ z \mid |z - z_0| \le \rho \}$ closed disk of radius ρ centered at z_0.

1 (Cauchy estimates) If f is analytic on $\overline{D}_\rho(z_0)$. Then if $|f(z)| \le M$ for $|z - z_0| = \rho$ then

$$|f^{(m)}(z_0)| \le \frac{m!}{\rho^m} M \quad \text{for any } m > 0.$$

← this estimate is useful in both proofs and applications..

It is surprising that the values of f at $|z - z_0| = \rho$ bounds the derivatives in the interior!

proof Recall the formula for $f^{(m)}(z_0)$ we got from the CIF:

$$f^{(m)}(z_0) = \frac{m!}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(z)}{(z - z_0)^{m+1}} dz$$

Let $z = z_0 + \rho e^{i\theta}$
$dz = i \rho e^{i\theta} d\theta$

$$\rightsquigarrow \quad \frac{1}{2\pi i} \frac{f(z)}{(z - z_0)^{m+1}} dz = \frac{f(z_0 + \rho e^{i\theta})}{\rho^m e^{im\theta}} \frac{d\theta}{2\pi}$$

$$\rightsquigarrow \quad f^{(m)}(z_0) = \frac{m!}{\rho^m} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) \underbrace{e^{-im\theta}}_{\le 1} \frac{d\theta}{2\pi}$$

Applying the M-L inequality:

$$|f^{(m)}(z_0)| \le \frac{m!}{\rho^m} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \frac{d\theta}{2\pi}$$

$$\le \frac{m!}{\rho^m} \int_0^{2\pi} M \frac{d\theta}{2\pi} = \frac{m!}{\rho^m} M. \qquad \square$$

This assumption is essential!

Theorem (Liouville's theorem) Suppose $f: \mathbb{C} \to \mathbb{C}$ is an analytic function (on all of $\mathbb{C}$!)
If f is bounded, then f is constant.

proof If f is bounded, there is an $M > 0$ s.t. $|f(z)| \le M$ for all $z \in \mathbb{C}$.

In particular, on $D_\rho(z_0)$, we have

$$|f'(z_0)| \le \frac{M}{\rho}.$$

Now let $\rho \to \infty$: $|f'(z_0)| = 0$ and so f is constant in a nbh of z_0 $\square$
$\longrightarrow$ f constant everywhere (since this holds for any z_0).

Defn An entire function f is a function f which is analytic on all of $\mathbb{C}$.

ex) $\left.\begin{array}{l} f(z) = a_n z^n + \dots + a_1 z + a_0 \\ f(z) = e^z, \cos z, \sin z, \dots \end{array}\right\}$ entire.

$\left.\begin{array}{l} \frac{1}{z} \\ \log z \\ \sqrt{z} \end{array}\right\}$ not entire.

Proof of the fundamental theorem of algebra: Every non-constant ! complex polynomial has a root in $\mathbb{C}$.

Suppose $p(z) \in \mathbb{C}[z]$ is a polynomial with no root. $\Rightarrow$ $\frac{1}{p(z)}$ is an entire function

It is also bounded: Suffices to show that $\frac{1}{p(z)} \to 0$ as $|z| \to \infty$.

$$\frac{p(z)}{z^n} = \frac{z^n + a_{n-1} z^{n-1} + \dots + a_0}{z^n} = 1 + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n} \xrightarrow[|z|\to\infty]{} 1$$

$\therefore$ $p(z) \sim z^n$ for $|z|$ large

$\therefore$ $p(z) \to \infty$ as $|z| \to \infty$

$\therefore$ $\frac{1}{p(z)} \to 0$ $\Rightarrow$ $\frac{1}{p(z)}$ bounded $\underset{\text{Liouville's theorem}}{\Rightarrow}$ $\frac{1}{p(z)}$ is constant

$\Rightarrow$ $p(z)$ constant $\square$

Morera's theorem

Thus, by Green's theorem $\int_R f(z)\,dz = 0$ for every closed rectangle in D.

Recall: We showed that $f(z)dz$ is a closed differential form $\Leftrightarrow$ f analytic.

Thm (Morera's theorem) f continuous on D. If

$$\int_R f(z)\,dz = 0$$

for every closed rectangle $R \subseteq D$ with sides parallel to the coordinate axes

Then f is analytic.

Rmk This is a "converse" to Cauchy's theorem. The amazing thing is that we require very little of f — only continuity! $\rightsquigarrow$ thm is useful for proving functions are analytic..

Complex notation (IV.8)

Define the differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right] \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right]$$

the 'average' of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial (iy)}$

f analytic

$$\Rightarrow \quad f'(z) = \frac{\partial f}{\partial x} \quad \text{and} \quad f'(z) = -i\frac{\partial f}{\partial y} = \frac{\partial f}{\partial (iy)} \qquad \therefore \quad f'(z) = \frac{\partial f}{\partial z}.$$

What about $\frac{\partial}{\partial \bar{z}}$? $\quad f = u + iv$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] + \frac{i}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]$$

CR eqs!

$\therefore$ f is analytic $\Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0.$