Power series

Aim: Showing that analytic functions f have power series expansions at every pt ← extremely useful fact!

$(a_k)_{k=0}^{\infty}$ sequence of complex numbers

$S_n = \sum_{k=0}^{n} a_k$ partial sums → new sequence $(S_n)_{n=0}^{\infty}$.

note: $a_k = S_k - S_{k-1}$ → can recover a_k from the partial sums

Defn "$\sum_{k=0}^{\infty} a_k$ converges" if the sequence (S_n) converges.

For a_k non-negative real numbers: S_n is monotonically non-decreasing ⇒ S_n converges iff it is bounded.

→ Thm (comparison test) $0 \le a_k \le r_k$ and $\sum r_k$ converges ⇒ $\sum a_k$ converges and $\sum a_k \le \sum r_k$.

Thm If $\sum a_k$ converges, ~~then~~ $a_k \to 0$ as $k \to \infty$.
($\sum a_k$ converges ⇔ S_n converges ⇔ $S_n - S_m \to 0$ as $m,n \to \infty$ ⇒ $a_k = S_n - S_{n-1} \to 0$)

Ex] $\sum_{k=0}^{\infty} z^k$ converges for all $|z|<1$:

$S_k = 1 + z + z^2 + \dots + z^k = \frac{1-z^{k+1}}{1-z} \xrightarrow[k\to\infty]{} \frac{1}{1-z}$

If $\sum a_k = A$ convergent, $\sum b_k = B$ convergent, then $\sum (a_k + b_k) = A + B$

Defn $\sum a_k$ converges absolutely if $\sum |a_k|$ converges.
← positive terms

This is stronger than convergence: $\sum \frac{(-1)}{n}$ converges, but not absolutely.

Thm $\sum a_k$ converges absolutely ⇒ $\sum a_k$ converges and $\left| \sum_{k=0}^{\infty} a_k \right| \le \sum_{k=0}^{\infty} |a_k|$

proof Write $a_k = x_k + i y_k$. We have

$|a_k| = \sqrt{x_k^2 + y_k^2} \ge \sqrt{x_k^2} = |x_k|$ also: $|a_k| \ge |y_k|$

∴ $\sum |a_k|$ converges ⇒ $\sum |x_k|$ converges, $\sum |y_k|$ converges (by comparison test)

⇒ $\sum x_k$ and $\sum y_k$ both converge ⇒ $\sum x_k + \sum i y_k = \sum a_k$ converges □

↑ the theorem for real series

in the real case..

Here is the proof that $\sum |x_n|$ converges $\Rightarrow$ $\sum x_k$ for $x_k \in \mathbb{R}$:

Define $p_n = \begin{cases} x_n & x_n \geq 0 \\ 0 & x_n < 0 \end{cases}$ $\quad q_n = \begin{cases} 0 & x_n \geq 0 \\ -x_n & x_n < 0 \end{cases}$

$\Rightarrow$ $p_0 + \dots + p_n \leq |x_0| + \dots + |x_n|$ $\Rightarrow$ $\sum p_n$ converges and $|\sum p_n| \leq \sum |x_n|$

$q_0 + \dots + q_n \leq |\cancel{x}_0| + \dots + |x_n|$ $\Rightarrow$ $\sum q_n$ converges

p_n, q_n both converge

then $\sum x_n = \sum (p_n - q_n) = \sum p_n - \sum q_n$ converges.

Finally, $x_0 + \dots + x_n \leq |x_0| + \dots + |x_n|$ $\Rightarrow \sum x_k \leq \sum |x_k|$. $\square$

ex) $\sum z^k$ converges absolutely, when $|z|<1$:

$$\sum |z|^k = \frac{1-|z|^k}{1-|z|\cancel{{}^k}} \underset{k\to\infty}{\to} 0$$

Also, $\cancel{\frac{1}{1-z}}$ $$\left|\frac{1}{1-z}\right| = \left|\sum z^k\right| \leq \left|\sum |z|^k\right| = \frac{1}{1-|z|} \qquad |z|<1$$

$$\therefore \left|\frac{1}{1-z} - \sum_{k=0}^{n} z^k\right| \leq \left|\sum_{k=n+1}^{\infty} z^k\right| = |z|^{n+1} \left|\sum_{k=0}^{\infty} z^k\right| \leq \frac{|z|^{n+1}}{1-|z|}.$$

Ex) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges (by the alternating series test) since the terms go to 0. But not absolutely:

$\sum_{k=1}^{\infty} \frac{1}{k}$ diverges:
$$S_n = \sum_{k=1}^{n} \frac{1}{k} \geq \sum_{k=0}^{n-1} \int_k^{k+1} \frac{1}{x}\,dx \qquad \left(\frac{1}{k} \geq \int_k^{k+1} \frac{dx}{x}\right.$$
$$= \int_1^{n} \frac{dx}{x} = \log n$$

$\therefore S_n \geq \log n \Rightarrow S_n$ diverges as $n \to \infty$.

Ex) $E = \mathbb{R}$

$\sum \frac{1}{k!} \frac{x^k}{1+x^{2k}}$ $\qquad g_k(x) = \frac{1}{k!} \frac{x^k}{1+x^{2k}}$ $\qquad \frac{1}{k!} \frac{|x|^k}{1+x^{2k}} \leq M_k$

Claim: This converges uniformly on all of $\mathbb{R}$.

Want to apply the M-test $\rightsquigarrow$ need M_k s.t. $|g_k(x)| \leq M_k \quad \forall k$

Should try to bound the fraction $\frac{|x|^k}{1+x^{2k}}$ (the factor $\frac{1}{k!}$ does the rest)

Note that for any $u \in \mathbb{R}$

$$\left| \frac{u}{1+u^2} \right| \leq \frac{1}{2}$$

similarly $\frac{2u}{1+u^2} \geq -1$

(this follows from $(u-1)^2 \geq 0 \Rightarrow 1+u^2 \geq 2u \Rightarrow 1 \geq \frac{2u}{1+u^2}$)

Take $u = x^k$:

$$\left| \frac{x^k}{1+x^{2k}} \right| \leq \frac{1}{2}$$

Take $M_k = \frac{1}{k!}$

Then $|g_k(x)| = \frac{1}{k!} \left| \frac{x^k}{1+x^{2k}} \right| \leq \frac{1}{2k!} < \frac{1}{k!}$

$\underset{\text{M-test}}{\Longrightarrow}$ $\sum \frac{1}{k!} \frac{x^k}{1+x^{2k}}$ converges uniformly (on $\mathbb{R}$).

Q: Is $\displaystyle\sum_{k=\text{[illegible]}}^{\infty} \frac{1}{k^2}\frac{z}{1+z^{2k}}$ an analytic function on $|z|<$

Not so obvious; we need to be able to say when the limit of a sequence of analytic functions is again analytic.

Sequences and series of functions

E : set

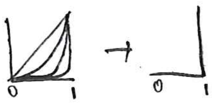
$f_j : E \to \mathbb{C}$: functions

Defn (f_j) converges pointwise on E if for each $x \in E$, $(f_j(x))$ converges; Set $f(x) = \lim_{j\to\infty} f_j(x)$. This gives a $\mathbb{C}$-valued function $f : E \to \mathbb{C}$ (which might not be continuous)

∴ a convergence notion for functions, this has some bad properties:

Ex1 $E = [0,1]$

$f_j(x) = x^j$



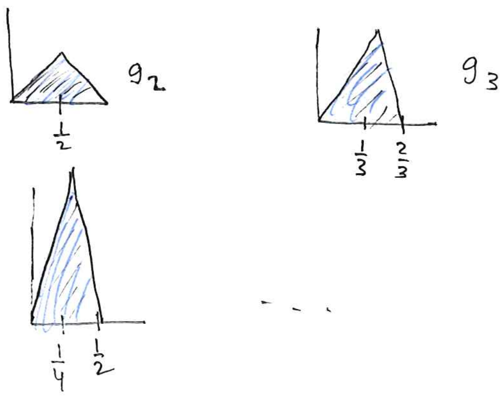
For $x \in [0,1]$ we get $f(x) = \lim_{j\to\infty} f_j(x) = \lim_{j\to\infty} x^j = \begin{cases} 0 & 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

∴ f_j are all continuous, but the limit function f is not:

$$0 = \lim_{x\to 1^-}\left(\lim_{n\to\infty} x^n\right) \ne \lim_{n\to\infty}\left(\lim_{x\to 1^-} x^n\right) = 1$$

Ex1

$$g_n(x) = \begin{cases} n^2 x & 0 \le x < \frac{1}{n} \\ 2n - n^2 x & \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & \frac{2}{n} \le x \le 1 \end{cases}$$



$$\int_0^{1/n} g_n(x)\,dx = \int_0^{1/n} n^2 x\,dx = \frac{n^2}{2}\cdot\left(\frac{1}{n}\right)^2 = \frac{1}{2}$$

$$\int_{1/n}^{2/n} g_n(x)\,dx = \int_0^{1/n} (2n - n^2 x)\,dx = 2\left(\frac{2}{n} - \frac{1}{n}\right) - \frac{n^2}{2}\left[\left(\frac{2}{n}\right)^2 - \left(\frac{1}{n}\right)^2\right] = 2 - \frac{3}{2} = 1$$

$$\therefore \int_0^1 g_n(x) = 1 \quad \forall n.$$

Now, let $n \to \infty$: $\lim_{n\to\infty} g_n(x) = 0$ for every $x \in (0,1]$ (since x eventually does not lie in $[0, \frac{2}{n}]$)

But:

$$1 = \lim_{n\to\infty} \int_0^1 g_n(x)\,dx \ne \int_0^1 \lim g_n(x)\,dx = 0$$

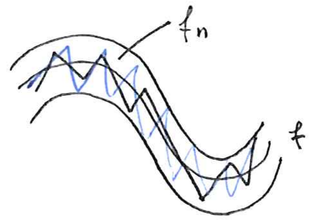
Pointwise
$\therefore$ Limits of analytic functions do not usually preserve continuity and integrals

$\rightsquigarrow$ need a stronger notion of convergence for functions ..

Defn (f_n) sequence of functions $f_n : E \to \mathbb{C}$
$f : E \to \mathbb{C}$ a $\mathbb{C}$-valued function.
(f_n) converges to f uniformly if for each $\varepsilon > 0$, there exists an N s.t

$$n > N \implies |f_n(x) - f(x)| < \varepsilon \quad \underline{\forall x \in E}$$

$\therefore$ The f_n are close to f everywhere :



Uniform convergence is a stronger form of convergence. It has the first desired property:

Thm (f_n) converging uniformly to f. If each f_n is continuous, then f is continuous.

proof. Let $\varepsilon > 0$, and let $a \in E$.
By uniform convergence: $\exists N$ s.t $\quad |f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $n > N$
By continuity of f_n at a: $\exists \delta > 0$ s.t

$$|f_n(z) - f(a)| < \frac{\varepsilon}{3} \quad \text{for all } |z-a| < \delta$$

Now we show that f is continuous at $z = a$:
If $|z - a| < \delta$, we have

$$|f(z) - f(a)| = |f(z) - f_n(z) + f_n(z) - f_n(a) + f_n(a) - f(a)|$$
$$\leq |f(z) - f_n(z)| + |f_n(z) - f_n(a)| + |f_n(a) - f(a)|$$
$$\leq \frac{\varepsilon}{3} + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

$\therefore$ f continuous at $z = a \implies f$ continuous on all of E $\square$

Thm $f_j(x)$ converges uniformly to f on E if and only if

$$\varepsilon_j = \sup_{x\in E} |f_j(x) - f(x)| \quad \text{converges to } 0 \text{ as } j\to\infty.$$

← "worst case estimator"

proof. ($\Rightarrow$): Let $\varepsilon>0$.

Want an N s.t. $\sup_{x\in E} |f_j(x)-f(x)| < \varepsilon$ for all $j>N$

Take $\varepsilon/2$ in the definition of uniform convergence:

$\exists N$ s.t. if $n>N$ then $|f_n(x) - f(x)| < \varepsilon/2 \quad \forall x\in E$

$$\Rightarrow \quad 0 \le \sup |f_n(x) - f(x)| \le \varepsilon/2 < \varepsilon$$

$\Rightarrow$ ok, by letting $\varepsilon\to 0$.

$\Leftarrow$: Let $\varepsilon>0$.

There is an N s.t. $n>N$ implies $\varepsilon_n = \sup_{x\in E} |f_n(x) - f(x)| < \varepsilon$

$$\Rightarrow \quad |f_n(x) - f(x)| \le \sup_{x\in E} |f_j(x) - f(x)| < \varepsilon \quad (\text{for all } x)$$

$\therefore$ $f_n \to f$ uniformly $\square$

Ex) $f_n(x) = x^n$, $f(x)=0$ on $E=(-1,1)$

$$\sup_E |f_n(x) - f(x)| = \sup_E |x^n| = 1$$

$\therefore$ does not tend to 0 $\longrightarrow$ not uniformly continuous.

Ex) $f_n(x) = x^n$, $f(x)=0$ on $E=(-r,r)$ $0<r<1$.

$$\sup |f_n(x) - f(x)| = \sup_E |x|^n \le r^n \longrightarrow 0 \quad \text{as } n\to\infty \Rightarrow \text{uniformly continuous}$$

Ex) For the "tent-functions" $g_n(x)$, we have

$$\varepsilon_n = \sup_{x\in[0,1]} |g_n(x) - 0| = n \quad \to \text{tends to } \infty!$$

Thm γ piecewise smooth curve in $\mathbb{C}$
f_j sequence of continuous $\mathbb{C}$-valued functions.
If $f_j \to f$ uniformly, then $\int_\gamma f_j(z)\,dz \longrightarrow \int_\gamma f(z)\,dz$

$\therefore$ If $f_j \to f$ uniformly, then the integrals also converge to $\int f$

proof.

$$\left| \int_\gamma f_j(z)\,dz - \int_\gamma f(z)\,dz \right| = \left| \int_\gamma \left(f_j(z) - f(z)\right) dz \right| \le \sup \underbrace{|f_j(z) - f(z)|}_{\to 0 \text{ as } j\to\infty} \cdot \text{length } \gamma \quad \square$$

$g_j(x)$ complex valued functions on E

Defn $s_n(x) = \sum_{j=0}^{n} g_j(x)$ partial sums

$\sum g_j(x)$ converges pointwise iff $(s_n(x))$ converges pointwise on E

$\sum g_j(x)$ converges uniformly iff $(s_n(x))$ converges uniformly on E.

Extremely useful theorem for checking uniform convergence:

Thm (Weierstrass M-test) $M_k \ge 0$ s.t $\sum M_k$ converges.
If $g_k(x)$ are $\mathbb{C}$-valued functions s.t $|g_k(x)| \le M_k \ \forall x \in E$.
Then $\sum g_k$ converges <u>uniformly</u> on E.

proof Let $z \in E$.
$|g_k(x)| \le M_k \Longrightarrow \sum g_k$ absolutely convergent and $\sum |g_k| \le \sum M_k$.

$$\Rightarrow \sum_{k=0}^{\infty} g_k(z) = g(z) \quad \text{for } g(z) \text{ a complex valued function on } E$$

$$\left(|g(z)| \le \sum |g_k(z)| \le \sum M_k \right)$$

Now,

$$|g(x) - s_n(x)| = \left| \sum_{k=n+1}^{\infty} g_k(x) \right| \le \underbrace{\sum_{k=n+1} M_k}_{\varepsilon_n}$$

and this converges to 0 as $k \to \infty$ since $\sum M_k$ converges..

$\Rightarrow \lim_{n\to\infty} |s_n(x) - g(x)| \to 0$ as $n \to \infty$ (indep of x)

$\Rightarrow s_n(x)$ converges uniformly to $g(x)$ on E.

$\square$

Thm $(f_k(z))$ sequence of analytic functions on a domain D
s.t. 在k阶· $f_k \to f$ uniformly. Then f is analytic on D.

proof. f_k analytic $\Rightarrow f_k$ continuous $\Rightarrow$ f is continuous (limit is uniform)
E = closed rectangle in D.

$$\int_{\partial E} f_k(z)\,dz = 0 \quad \forall k \quad \text{since } f_k \text{ is analytic (Cauchy's theorem)}$$

$$\Rightarrow \int_{\partial E} f(z)\,dz = \int_{\partial E} \lim f_k(z)\,dz = \lim \int_{\partial E} f_k(z)\,dz = 0 \quad \Rightarrow \int_{\partial E} f(z)\,dz$$

$\Rightarrow$ f is analytic, by Morera's theorem. $\square$

What happens to the derivatives $f_k^{(m)}(z)$?

Thm $f_k(z)$ analytic for $|z - z_0| \le R$
$f_k \to f$ uniformly on —||—
Then for all $r < R$, $m \geq 1$ $\quad f_k^{m}(z) \longrightarrow f^{(m)}(z)$ uniformly on $|z - z_0| \le r$.

proof. Let $\varepsilon_k = \sup\limits_{|z - z_0| < R} |f_k(z) - f(z)|$
By assumption $\varepsilon_k \to 0$ as $k \to \infty$.
Fix $s \in (r, R)$.
By CIF:

$$f_k^{(m)}(z) - f^{(m)}(z) = \frac{m!}{2\pi i} \int_{|z - z_0| = s} \frac{f_k(w) - f(w)}{(w - z)^{m+1}}\,dz$$

Note: $\begin{array}{l} |w - z_0| = s \\ |z - z_0| \le r \end{array} \Rightarrow |w - z| \geq s - r > 0$

$$\therefore \left| \frac{f_k(w) - f(w)}{(w - z)^{m+1}} \right| \le \frac{\varepsilon_k}{(s - r)^{m+1}}$$

ML estimate:

$$\left| f_k^{(m)}(z) - f^{(m)}(z) \right| = \left| \frac{m!}{2\pi i} \int_{|z - z_0| = s} \frac{f_k(w) - f(w)}{(w - z)^{m+1}}\,dz \right| \le \frac{m!}{\cancel{2\pi}} \frac{\varepsilon_k}{(s - r)^{m+1}} \cdot \cancel{2\pi} s = \frac{m!}{(s - r)^{m+1}} \cdot \varepsilon_k$$

$\longrightarrow 0$ as $k \to \infty$.

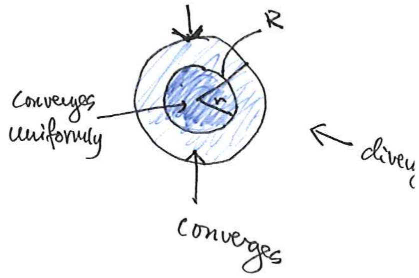
Power series

<u>Defn</u> A <u>power series</u> is a series of the form

$$\sum_{k=0}^{\infty} a_k (z-z_0)^k. \quad \leftarrow$$

usually we take $z_0 = 0$. This is wlog, since we can always perform a change of variables.

<u>Thm</u> $\sum a_k z^k$ power series.
Then $\exists R \in [0,\infty]$ s.t $\sum a_k z^k$ converges absolutely for every $|z| < R$ and diverges for $|z| > R$. For each $r < R$, $\sum a_k z^k$ converges <u>uniformly</u> for $|z| \le r$.



R is called the <u>radius of convergence</u> of $\sum a_k z^k$.
Note that R can be 0 or ∞:

<u>ex1</u> $\sum n! z^n \Rightarrow R = 0$

<u>ex1</u> $\sum \frac{z^n}{n!} \Rightarrow R = \infty \quad \leftarrow$ prove these!

<u>ex1</u> $\sum z^n \Rightarrow R = 1$

<u>ex1</u> $\sum 2^m (z-2)^m \Rightarrow R = 1/2$

$2^m (z-2)^m = (2z-4)^m \leftarrow$ want this to be bounded $\Rightarrow |2z-4| < 1$

$\Rightarrow |2z-4| < 1 \Rightarrow \frac{3}{2} < z < \frac{5}{2}$

<u>proof of theorem</u>

If $(|a_k| r_0^k)$ is bounded for some $r = r_0 \Rightarrow (|a_k| r^k)$ bounded for all $r < r_0$

$R = \sup \{ r \mid \underbrace{|a_k| r^k}_{\text{bonded}} \} \leftarrow$ might be $+\infty$!

$\therefore |a_k| r^k$ bonded for all $r < R$ (but $|a_k| R^k$ might be unbonded).

Also $|a_k| s^k$ is <u>unbonded</u> for all $s > R$ (by defintion of R)

$\Rightarrow \sum a_k z^k$ <u>diverges</u> for $|z| > R$.

If $r < R$, then choose s s.t $r < s < R$.

$|a_k| s^k$ bonded $\Rightarrow$ let C be a constant s.t $|a_k| s^k \le C$ for all k

$\therefore |a_k z^k| \le |a_k| r^k = |a_k| s^k \left(\frac{r}{s}\right)^k \le C \cdot \left(\frac{r}{s}\right)^k$

$\leftarrow$ this number is < 1.

$\therefore$ let $M_k = C\left(\frac{r}{s}\right)^k$, apply Weierstrass M-test $\Rightarrow \sum a_k z^k$ converges uniformly $\square$

~~Thm~~ let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ where $\sum a_k z^k$ has a positive (or ∞) radius of convergence R.

Then $f(z)$ is analytic.

Formulas for the derivatives:

$$f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1} \qquad \text{for} \quad |z| < R$$

$$f''(z) = \sum_{k=2}^{\infty} k(k-1) z^{k-2}$$

$$\vdots$$

$f(z) = \sum_{k=0}^{\infty} a_k z^k$ coincides with its Taylor series: $a_k = \dfrac{f^{(k)}(0)}{k!}$ $k \geq$

proof If $\sum_{k=0}^{n} a_k z^k$ ~~is~~ uniformly convergent to $f(z)$ then f is analytic (being a uniform limit of (analytic) polynomials).

For the formula for a_k:

$$f(0) = a_0 + a_1 z + \cdots \Big|_{z=0} = a_0 \quad \checkmark$$

$$f^{(k)}(0) = \sum_{j=k}^{\infty} j(j-1)\cdots(j-k+1)\, a_j\, z^{j-k} \Big|_{z=0}$$

$$= k!\, a_k \qquad \therefore \quad a_k = \frac{f^{(k)}(0)}{k!} \qquad \square$$

Slightly more generally, if $f(z) = \sum a_k (z - z_0)^k$, then $a_k = \dfrac{f^{(k)}(z_0)}{k!}$

Convergence tests

Thm (Ratio test) If $\lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right| = A \le \infty$ (including ∞!)

Then the radius of convergence of $\sum a_k z^k$ is A.

Thm (Root test) If $\sqrt[k]{|a_k|}$ has a limit as $k\to\infty$ (including ∞!)

Then the radius of convergence of $\sum a_k z^k$ is $R = \frac{1}{\lim \sqrt[k]{|a_k|}}$

Proof (Ratio test) $A = \lim \left|\frac{a_k}{a_{k+1}}\right|$. If $r < A$, then $\left|\frac{a_k}{a_{k+1}}\right| > r$ for large k, say for $k \geq N$.

$\Rightarrow |a_k| > r|a_{k+1}| \quad k \geq N$

$\Rightarrow |a_N| r^N \geq |a_{N+1}| r^{N+1} \geq |a_{N+2}| r^{N+2} \geq \dots$

$\Rightarrow (|a_k| r^k)$ bounded

$\Rightarrow R \geq r$ (by defn of R)

$\Rightarrow R \geq A$ (since we can let $r \to A$)

If $s > A$, then $\left|\frac{a_k}{a_{k+1}}\right| < s$ for large k, say $k \geq N$

$\Rightarrow |a_k| < s|a_{k+1}| \quad k \geq N$

$\Rightarrow |a_N| s^N \leq |a_{N+1}| s^{N+1} \leq |a_{N+2}| s^{N+2} \leq \dots$

$\Rightarrow a_k z^k$ does not tend to 0 for $|z| \geq s \Rightarrow$ diverges

$\Rightarrow s \geq R$

$\Rightarrow \cancel{L} \geq R$ (since we can let $s \to A$)

$\therefore L = R$.

(Root test): $\cancel{A} = \lim \cancel{a_k}$ $\frac{1}{\lim \sqrt[k]{|a_k|}}$. If $r > A \Rightarrow \sqrt[k]{|a_k|}\, r > 1$ for large k ($k > N$

$\Rightarrow a_k z^k$ does not tend to 0 for $|z| = r$ and $r \geq R$

On the other hand, if $r < A$, $|a_k| r^k < 1$ is bounded $\Rightarrow r \leq R$ (by def of R!)

$\Rightarrow R = A$. ∎

ex) $\sum a_k z^k$ where $a_k = \dfrac{k}{6^k}$

$$\Rightarrow \quad \frac{a_k}{a_{k+1}} = \frac{\frac{k}{6^k}}{\frac{k+1}{6^{k+1}}} = 6 \cdot \frac{k}{k+1} \xrightarrow[k\to\infty]{} 6 \qquad \therefore R = 6$$

Alternatively, $R = 1/\lim \sqrt[k]{k/6^k}$

$$\lim \sqrt[k]{\frac{k}{6^k}} = \lim \frac{k^{1/k}}{6} = \frac{1}{6}$$

$$\therefore R = \frac{1}{1/6} = \underline{6}.$$

Some computations of power series

ex) $f(z) = \dfrac{1}{(1-z)^2}$, for $|z|<1$.

Note that $f(z) = \dfrac{\partial}{\partial z} F(z)$ where $F(z) = \dfrac{1}{1-z}$. Now, $F(z) = 1 + z + z^2 + \ldots$ for $|z|<1$

$$F(z) = \sum z^k \Rightarrow F'(z) = \sum_{k=1} k z^{k-1} = \sum_{k=0}^{\infty} (k+1) z^k.$$

Ex) $-\mathrm{Log}(1-z)$, for $|z|<1$.

Let $f(z) = -\mathrm{Log}(1-z)$. We have $f(z) = \displaystyle\int_0^z \frac{dw}{1-w}$

$$= \int_0^z \sum_{k=0}^{\infty} w^k \, dw$$

uniform convergence implies that we can switch the order of $\int$ and Σ.

$$= \sum_{k=0}^{\infty} \frac{z^{k+1}}{k+1} = z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \ldots$$

$$\therefore \mathrm{Log}\, w = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (w-1)^k.$$ ← valid for $|w-1|<1$

Ex) $f(z) = \dfrac{e^z}{1-z}$

$$= \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\right)\left(1 + z + z^2 + z^3 + \ldots\right)$$

$$= 1 + z\left(1 + \frac{1}{1!}\right) + z^2\left(1 + \frac{1}{1!} + \frac{1}{2!}\right) + z^3\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}\right) + \ldots$$

$$= 1 + 2z + \frac{5}{2} z^2 + \frac{8}{3} z^3 + \ldots$$

The new power series $f \cdot g$ has convergence radius ~~at least~~ $\geq \min(R_f, R_g)$

More generally, if $f = \sum a_k z^k$, $g = \sum b_k z^k$ are two power series $\to fg = \sum c_k z^k$

where $c_k = \displaystyle\sum_{p+q=k} a_p b_q = \sum_{j=0}^{k} a_j b_{k-j}$ ← "Cauchy product"

Can also form $\dfrac{f(z)}{g(z)}$ by expanding $\dfrac{1}{g(z)}$: Assume $g(0)=1$ then

$$\frac{1}{g(z)} = \frac{1}{1 + \sum_{k=1}^{\infty} b_k z^k} = \sum_{n=0}^{\infty} (-1)^n \left(\sum_{k=0}^{\infty} b_k z^k\right)^n$$

ex) $f(z) = \dfrac{\sin z}{\cos z}$

$$\frac{1}{\cos z} = \frac{1}{1 - \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \ldots\right)} = 1 + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \ldots\right) + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \ldots\right)^2 + \ldots$$

$$= 1 + \frac{1}{2} z^2 + \frac{5}{24} z^4 + \ldots \qquad \left(\frac{1}{4} - \frac{1}{24} = \frac{5}{24}\right)$$

$$\Rightarrow f(z) = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)\left(1 + \frac{1}{2} z^2 + \frac{5}{24} z^4 + \ldots\right) = z - \frac{1}{3} z^3 + \frac{2}{15} z^5$$

Power series expansions of analytic functions

Have seen that (convergent) power series $\sum_{k=0}^{\infty} a_k z^k$ are analytic inside of their domain of convergence. Now, we will prove the converse: that every analytic function $f(z)$ has a convergent power series expansion around every point.

⇝ Most questions about analytic functions are reduced to questions about power series (and hence polynomials).

Thm $f(z)$ analytic for $|z-z_0|<\rho$

$\Rightarrow$ f is represented by a power series

$$f(z)=\sum_{k=0}^{\infty} a_k (z-z_0)^k \qquad \text{on} \qquad |z-z_0|<\rho$$

where

$$a_k = \frac{f^{(k)}(z_0)}{k!} \qquad k \geq 0$$

The radius of convergence R is $\geq \rho$, and for any $0<r<\rho$ we have

$$a_k = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w-z)^{k+1}}\, dw \qquad k \geq 0$$

Futhermore, if $|f(z)| \leq M$ for all $|z-z_0|=r$, then

$$|a_k| \leq \frac{M}{r^k}, \qquad k \geq 0.$$

- f has a power series expansion on all of $|z-z_0|<\rho$
- We have explicit formulas and bounds for the coefficients.

proof. Given f, z_0, ρ as above. $\rightsquigarrow$ can assume $z_0 = 0$ by change of variables

Let $z \in \mathbb{C}$ $|z| < r$. $(r < \rho)$

For w s.t. $|w| = r$:

$$\frac{1}{w-z} = \frac{1}{w} \cdot \frac{1}{1 - z/w} = \frac{1}{w} \sum_{k=0}^{\infty} \left(\frac{z}{w}\right)^k = \sum_{k=0}^{\infty} \frac{z^k}{w^{k+1}}$$

has $|\ | < 1$ $\longrightarrow$ the series converges uniformly on $|z| \le r$.

$$\therefore f(z) = \frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{w-z} dw$$

$$= \frac{1}{2\pi i} \int_{|w|=r} f(w) \left(\sum_{k=0}^{\infty} \frac{z^k}{w^{k+1}} \right) dw$$

$$= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \left(\int_{|w|=r} \frac{f(w)}{w^{k+1}} dw \right) z^k$$

$$= \sum_{k=0}^{\infty} a_k z^k \quad \text{where} \quad a_k = \frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{(w-z)^{k+1}} dw$$

when $r < \rho$, this coincides with $\frac{f^{(k)}(z_0)}{k!}$ (Taylor expansion).

This series converges on $|z| < r$ $\Rightarrow$ $R \geq \rho$ (by letting $r \to \rho$)

The bounds on $|a_k|$ follow from the Cauchy estimates. $\square$

A very useful corollary:

Cor If $f(z)$, $g(z)$ analytic for $|z - z_0| < r$, and $f^{(k)}(z_0) = g^{(k)}(z_0)$ $\forall k \geq 0$

Then $f(z) = g(z)$ (for $|z - z_0| < r$)

$\therefore$ the function f is defined by the values of its derivatives at a single point!

or f analytic at z_0, $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^k$ power series expansion

R = radius of convergence $= \sup \{ R_0 \mid f$ extends to an analytic function on $|z - z_0| < R_0 \}$

$\therefore$ Can detect R from the distance to the nearest point where f is not analytic

Some classical examples

1) $f(z) = e^z$ $\qquad a_k = \frac{1}{k!} \underbrace{\frac{d^k}{dz^k} e^z \Big|_{z=0}}_{e^z|_{z=0}} = \frac{1}{k!}$

$$\therefore \; e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

e^z is entire $\Rightarrow$ $R = +\infty$.

not hard to check (using eg the ratio test) that this series converges on all of $\mathbb{C}$.

2) $$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

$$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$$

can be deduced from the identities $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \frac{e^{iz} + e^{-iz}}{2}$ or by direct differentiation..

3) $$f(z) = \frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \dots$$

Converges in $|z| < 1$. Q: Why not all of $\mathbb{C}$?
A: $f(z)$ is not analytic at $z = \pm i \rightsquigarrow R = 1$.

The complex numbers give better insight to the real function $\frac{1}{1+x^2}$!

4) $f(z) = \frac{z^3 - 1}{z^2 - 1}$ $\qquad$ Around $z = 2$: $\quad f(z) = \sum_{k=0}^{\infty} a_k (z-2)^k$

At $z = 1$ we can 'remove' the singularity: $f(z) = \frac{\cancel{(z-1)}(z^2+z+1)}{\cancel{(z-1)}(z+1)} = \frac{z^2+z+1}{z+1}$

$\rightsquigarrow f(z)$ still has a singularity at $z = -1$.

$\therefore R = 3$ since the function can be extended to all of $\{|z-2| < 3\}$.

Indeed,
$$f(z) = (z^2+z+1)\frac{1}{z+1} = (z^2+z+1)\frac{1}{3 + (z-2)}$$
$$= (z^2+z+1)\frac{1/3}{1 + \frac{z-2}{3}} = \frac{z^2+z+1}{3} \sum_{k=0}^{\infty} \frac{(-1)^k}{3^k}(z-2)^k$$

and this visibly has $R = 3$.

(5) $f(z) = \dfrac{\operatorname{Log} z}{z-1}$ near $z=5$. $f(z) = \sum_{k=0}^{\infty} a_k (z-5)^k$

Again at $z=1$ we can 'cancel' the singularities of $z=1$ and $\operatorname{Log} z$:

$$\frac{\operatorname{Log} z}{z-1} = 1 - \frac{1}{2}(z-1) + \frac{1}{3}(z-1)^2 - \frac{1}{4}(z-1)^3 + \dots$$

$\therefore \lim_{z \to 1} f(z) = 1$.

However $z=0$ is a problem: $\operatorname{Log} 0$ is not defined.

Hence the radius of convergence of $\sum a_k (z-5)^k$ is $\underline{R=5}$.

Power series expansions at $z=\infty$.

Defn $f(z)$ is analytic at ∞ if the function

$$g(w) = f\left(\frac{1}{w}\right)$$

is analytic at $w=0$.

ex) $f(z) = \dfrac{az+b}{cz+d}$ analytic at ∞ $\Leftrightarrow$ $\dfrac{aw^{-1}+b}{cw^{-1}+d}$ analytic at $w=0$

$ad-bc \neq 0$

$\Leftrightarrow$ $\dfrac{a+bw}{\cancel{cw}\,c+dw}$ analytic at $w=0$

$\Leftrightarrow$ $c \neq 0$

ex) $f(z) = \dfrac{1}{z^n}$ $\rightsquigarrow$ $g(w) = w^n$ $\longrightarrow$ analytic at $w=0$ ✓

ex) $f(z) = \dfrac{1}{z^2+1}$ $\rightsquigarrow$ $g(w) = \dfrac{1}{w^{-2}+1} = \dfrac{w^2}{w^2+1}$ $\longrightarrow$ ok.

$g(w)$ analytic at 0 $\rightsquigarrow$ power series expansion at $w=0$

$$g(w) = \sum_{k=0}^{\infty} b_k w^k = b_0 + b_1 w + b_2 w^2 + \ldots \qquad |w| < \rho$$

$$\therefore\ f(z) = \sum_{k=0}^{\infty} \frac{b_k}{z^k} = b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3} + \ldots \qquad \text{for } |z| > \frac{1}{\rho}$$

converges absolutely for $|z| > \frac{1}{\rho}$

uniformly for $|z| > \frac{1}{r}$ for any $r < \rho$.

How to find b_k in terms of f?

Note:

For $r > \frac{1}{\rho}$:

$$\int_{|z|=r} f(z) z^m \, dz = \int_{|z|=r} \left(\sum_{k=0}^{\infty} b_k z^{-k}\right) dz$$

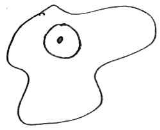
uniform convergence → allowed to switch Σ and $\int$

$$\overset{\curvearrowright}{=} \sum_{k=0}^{\infty} b_k \underbrace{\left(\int_{|z|=r} z^{m-k} \, dz\right)}_{= \begin{cases} 2\pi i & \text{if } m-k=-1 \\ 0 & \text{otherwise} \end{cases}} = \underline{2\pi i \, b_{m+1}}$$

$$\therefore \quad b_k = \frac{1}{2\pi i} \int_{|z|=r} f(z)\, z^{k-1}\, dz \qquad k \geq 0$$

We will see a more general formula later when we discuss Laurent expansions.

Recap: $z_0 \in D$



$f: D \to \mathbb{C}$ a function.

$f(z)$ analytic at $z = z_0 \Leftrightarrow f(z)$ has a uniformly convergent power series at $z = z_0$

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad a_k = \frac{1}{2\pi i} \int \frac{f(w)}{(w - z_0)^{k+1}} dw$$

Also, if f is analytic at $z = \infty$: $f(z) = b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots \leftarrow$ uniformly convergent in $|z| > r$

$\uparrow$

$g(w) = f(\frac{1}{w})$ analytic at $w = 0$

e.g. $\frac{z+1}{z^2} \Rightarrow g = \frac{w^{-1}+1}{w^{-2}} = w + w^2$

Went on to talk about zeros of analytic functions

f has a zero of order N at $z = a$ if
$f(a) = f'(a) = \dots = f^{(N-1)}(a) = 0$
$f^{(N)}(a) \neq 0$

$$\Leftrightarrow \quad f(z) = a_N (z-a)^N + a_{N+1}(z-a)^{N+1} + \dots$$
$a_N \neq 0$.

$$\Leftrightarrow \quad f(z) = (z-a)^N h(z) \qquad h \text{ analytic and } h(a) \neq 0$$

Key to finding the power series expansion:

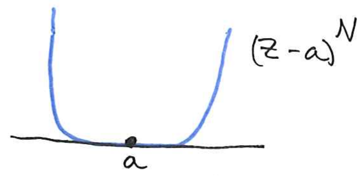
$$\frac{1}{w-z} = \frac{1}{w} \frac{1}{1 - \frac{z}{w}} = \sum_{k=0}^{\infty} \frac{z^k}{w^{k+1}}$$

$$\rightsquigarrow \frac{1}{2\pi i} \int \frac{f(w)}{w-z} dw = \dots = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \underbrace{\left(\int_{|w|=r} \frac{f(w)}{w^{k+1}} dw \right)}_{= \frac{f^{(k)}(0)}{k!}} z^k$$

Zeros of analytic functions

$a \in \mathbb{C}$

$f(z)$ analytic at $z = a$.



the order is a measure on how fast f decreases to 0 as $z \to a$.

Defn f has a zero of order N at $z=a$ if

$$f(a) = f'(a) = f^{(2)}(a) = \dots = f^{(N-1)}(a) = 0$$
$$f^{N}(a) \neq 0$$

$\Leftrightarrow$ f has a power series expension at $z=a$ of the form

$$f(z) = a_N (z-a)^N + a_{N+1}(z-a)^{N+1} + \dots \qquad \boxed{a_N \neq 0}$$

$\Leftrightarrow$ $f(z) = (z-a)^N h(z)$ for some analytic function $h(z)$, $h(a) \neq 0$.
(so N is the largest power with this property).

Let $\mathrm{ord}_a(f)$ denote this N (note that $\mathrm{ord}_a(f) = 0 \Leftrightarrow f(a) \neq 0$)

$\mathrm{ord}_a(f) = 1 \rightsquigarrow$ we say f has a simple zero at $z=a$

$\mathrm{ord}_a(f) = 2 \rightsquigarrow$ —— " —— double zero at $z=a$.

$\rightsquigarrow$ this is a natural integer to consid e.g if $P(z) = \prod(z$ then $\mathrm{ord}_{a_i} P = n_i$
$\rightsquigarrow$ ord is the "exponent of $(z-a_i)$"

$\mathrm{ord}_a(f)$ satisfies the following properties:

(i) $\mathrm{ord}_a(f \cdot g) = \mathrm{ord}_a(f) + \mathrm{ord}_a(g)$

(ii) $\mathrm{ord}_a(f+g) \geq \min(\mathrm{ord}_a(f), \mathrm{ord}_a(g))$

$\leftarrow$ to see this:
$f(z) = (z-a)^m h_1(z)$
$g(z) = (z-a)^n h_2(z)$
if $m \leq n$:
$f(z)\cdot g(z) = (z-a)^{m+n} h_1 h_2$
$f(z) + g(z) = (z-a)^m \underbrace{(h_1 + (z-a)^{n-m} h}_{\text{analytic}}$

This also makes sense for $a = \infty$:

$$\mathrm{ord}_\infty(f) = \mathrm{ord}_0(g)$$
where $g(w) = f\left(\frac{1}{w}\right)$

$$\therefore \ \mathrm{ord}_\infty(f(z)) = N \Leftrightarrow f(z) = \frac{b_N}{z^N} + \frac{b_{N+1}}{z^{N+1}} + \dots \qquad b_N \neq 0$$

ex $f(z) = z^N$ has a zero of order N at $z=0$.

$$\therefore \ \mathrm{ord}_a(f) = \begin{cases} N & a = 0 \\ 0 & \text{otherwise} \end{cases}$$

$\therefore \quad \operatorname{ord}_\infty(f) = n \iff f(z) = \dfrac{b_n}{z^n} + \dfrac{b_{n+1}}{z^{n+1}} + \cdots \qquad b_n \neq 0.$

<u>ex</u> $f(z) = \dfrac{1}{1+z^2}$ has $\operatorname{ord}_\infty(f) = 2$: A double zero at $z = \infty$

$g(w) = f(1/w) = \dfrac{w^2}{1+w^2} = w^2 \cdot$ holomorphic function

<u>Defn</u> Let $E \subset \mathbb{C}$ be a set.

$a \in E$ is <u>isolated</u> if there exist a disk $D \subset \mathbb{C}$ s.t

$$D \cap E = \{a\}.$$

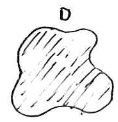
$\iff \exists \varepsilon > 0$ s.t $|z - a| > \varepsilon \ \forall z \in E - \{a\}$.

We say E is <u>isolated</u>, if every $a \in E$ is isolated.

<u>ex</u>

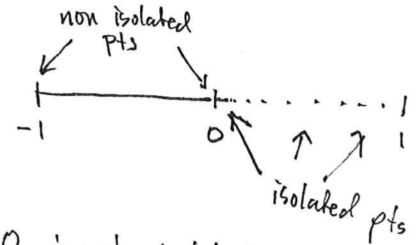
finite set of pts $\rightarrow$ isolated

<u>ex</u>



D domain $\Rightarrow$ not isolated. (none of the points are isolated)

<u>ex</u>

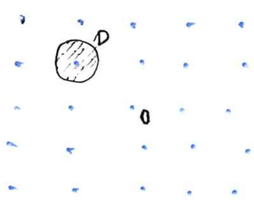


$E = [-1, 0] \cup \{ \frac{1}{n} \mid n \geq 1 \}$

0 is not isolated, since any disk $D \ni 0$ will contain negative real numbers.

$\frac{1}{2}$ is isolated, since $D = \{ |z - \frac{1}{2}| < \frac{1}{7} \}$ avoids all other $\frac{1}{n}$

<u>ex</u>



$E = \{ m + ni \mid m, n \in \mathbb{Z} \}$

$\Rightarrow E$ is isolated

Thm $f : D \to \mathbb{C}$ analytic (D a domain)

Assume $f \not\equiv 0$, then the zeros of f are isolated.
(f not the zero function)

$\therefore$ cannot have a convergent sequence of zeroes of f !

Proof Define

$$U = \left\{ z \in D \;\middle|\; f^{(m)}(z) = 0 \text{ for all } m \geq 0 \right\}$$

$a \in U \Rightarrow f$ has a power series expansion $\sum a_k (z-a)^k$ where $a_k = \dfrac{f^{(k)}(a)}{k!} = 0$

$\Rightarrow f(z) = 0$ for all z in a disk $D_\rho(a)$

Note: $D_\rho(a) \subseteq U$ (because all derivatives vanish for the zero function)

$\Rightarrow D$ is an open set.

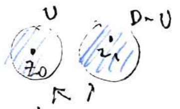
If $a \in D \setminus U$ then $f^{(k)}(a) \neq 0$ for some $k \geq 0$ (by def of U

$\Rightarrow f^{(k)}(z) \neq 0$ for all z in a disk $D_\rho(a)$ $\rho > 0$

$\Rightarrow D - U$ also open { Note: $D = U \cup (D - U)$) open ?? + disjoint

$\Rightarrow$ either $D = U$ or $U = \emptyset$ $\Rightarrow U = \emptyset$.

D a domain
$\Rightarrow D$ has no ~~proper~~ disjoint subsets which are both open (see Exc. II.1.18)



cannot connect z_0 and z_1!

$D \neq U$: consider $m=1$
if $D=U$, then f is the zero function!

$\Rightarrow$ Every zero of f is of finite order.

If a is a zero of order $N = \text{ord}_a(f)$, then

$f(z) = (z-a)^N h(z)$ where h is analytic near a + $h(a) \neq 0$.

$\therefore$ For $\varepsilon > 0$ very small $h(z) \neq 0$ on $D_\varepsilon(a)$

$\therefore$ $|f(z)| \neq 0$ for $0 < |z-a| < \varepsilon$

$\Rightarrow a$ is an isolated pt. $\square$

Theorem (Uniqueness principle) $f : D \to \mathbb{C}$, $g : D \to \mathbb{C}$ analytic (D is a domain)

If $f(z) = g(z)$ for all $z \in E$ where E is a set with a non-isolated pt

then $f = g$.

proof Let $h(z) = f(z) - g(z)$.

$h(z) = 0 \quad \forall z \in E \Rightarrow h(z) = 0$ for all $z \in D$ (by the previous theorem)

$\Rightarrow f = g$.

Ex Show that

$$\cos^2 z + \sin^2 z = 1 \qquad \forall z \in \mathbb{C}$$

Solution:

Let $f(z) = \cos^2 z + \sin^2 z$

$g(z) = 1$

$E = \mathbb{R}$

$f(z) = g(z) \quad \forall z \in \mathbb{R}$

$\Rightarrow f(z) = g(z) \quad \forall z \in \mathbb{C}$

$\Rightarrow \cos^2 z + \sin^2 z = 1 \quad \forall z \in \mathbb{C}$ ✓

Another example: $\tan^2 z = \sec^2 z - 1$ holds for all $z \in \mathbb{C}$ (because it holds on $\mathbb{R}$)

Ex Note that $e^{z} e^{w} = e^{z+w}$ for all $z, w \in \mathbb{R}$ $\Rightarrow$ it also holds for all $z, w \in \mathbb{C}$ (fix each variable separately and apply the uniqueness theorem)

Ex Analytic functions can have infinitely many zeros:

$f(z) = \sin(\pi z) = 0$ for all $z = 0, \pm 1, \pm 2, \ldots$

Consider $f(z) = \sin\left(\frac{\pi}{z}\right)$ $\quad f\left(\frac{1}{n}\right) = \sin\left(\frac{\pi}{\frac{1}{n}}\right) = \sin(\pi n) = 0$

$\therefore$ f is 0 on $E = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$ $\leftarrow$ this has accumulation pt!

There is no contradiction however, since f is not analytic at $z = 0$.