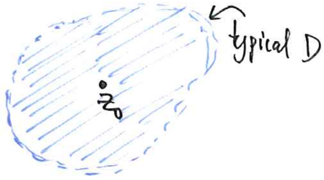
Laurent series and isolated singularities

What we have done:

$f: D \to \mathbb{C}$ analytic



Near z_0:

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

$\therefore$ analytic functions = power series

only non-negative powers

Where we are going:

$f: D \to \mathbb{C}$ analytic

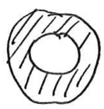
allow more general functions: $f(z) = \frac{1}{z-z_0}$

more general domains: z_0 might be outside D.

Near z_0;

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)$$

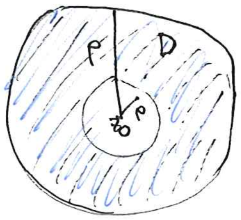
negative powers

$\therefore$ replace  $|z| < \rho$ with this:  $\sigma < |z| < \rho$

$a_0 + a_1 z + a_2 z^2 + \ldots$ with this: $\ldots + a_{-1} z^{-1} + a_0 + a_1 z + a_2 z^2 + \ldots$

Laurent Series and isolated singularities

Let $D = \{ z \mid \rho < |z - z_0| < \sigma \}$ $\boxed{0 \le \rho < \sigma \le \infty}$



$f : D \to \mathbb{C}$ analytic function

Thm f can be decomposed as

$$f(z) = f_0(z) + f_1(z)$$

this is useful because the domains here are simpler (they are disks or complements of disks).

where
- $f_0(z)$ analytic in $|z - z_0| < \sigma$
- $f_1(z)$ analytic in $|z - z_0| > \rho$

ex| $f(z) = \frac{z^3 + z + 1}{z} = \underbrace{z^2 + 1}_{f_0(z)} + \underbrace{\frac{1}{z}}_{f_1(z)}$

$\rho = 0$
$\sigma = \infty$

This decomposition is unique if we normalize s.t $f_1(\infty) = 0$.

ex| $f(z) = \frac{1}{(z-1)(z-2)}$ is analytic on $D = \{ 1 < |z| < 2 \}$

$f_0 = \frac{1}{z-2} = \frac{-1/2}{1 - z/2} = -\frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^k$

or, since $f'(z) = \frac{-1}{(z-2)}$

We find its decomposition by partial fractions decomposition:

$$\frac{1}{(z-1)(z-2)} = \underbrace{\frac{1}{z-2}}_{f_0(z)} - \underbrace{\frac{1}{z-1}}_{f_1(z)}$$

f_0 analytic for $|z| < 2$
f_1 analytic for $|z| > 1$
and $f_1(\infty) = 0$.

Remarks: If f is analytic on $|z - z_0| < \sigma \Rightarrow f(z) = f_0(z)$ $f_1(z) = 0$
If f is analytic on $|z - z_0| > \rho$ and $f(\infty) = 0$, then $f = f_1$ and $f_0 = 0$.

Proof ① Uniqueness: If $f(z) = f_0(z) + f_1(z) = g_0(z) + g_1(z)$ with $f_1(\infty) = g_1(\infty) = 0$.

$\Rightarrow g_0(z) - f_0(z) = f_1(z) - g_1(z)$ on $\rho < |z - z_0| < \sigma$

Define $h(z) = \begin{cases} g_0(z) - f_0(z) & |z - z_0| < \sigma \\ f_1(z) - g_1(z) & |z - z_0| > \rho \end{cases}$ these agree on the overlap at $\underbrace{\rho < |z - z_0| < \sigma}_{D}$!

$\therefore h$ is an entire function, $h(z) \underset{z \to \infty}{\to} 0$

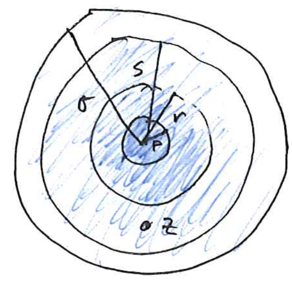
$\underset{\text{Liouville}}{\Rightarrow}$ $h = $ constant $\Rightarrow h = 0$ since the constant is 0 on D $\Rightarrow g_0(z) = f_0(z)$ and $g_1(z) = f_1(z)$ $\forall z$

$\Rightarrow$ uniqueness OK.

② Existence

Using (once again) Cauchy's integral formula:

Pick r, s s.t $\rho < r < s < \sigma$.



$$f(z) = \underbrace{\frac{1}{2\pi i}\int_{|w-z_0|=s} \frac{f(w)}{(w-z)}\,dw}_{=:\, f_0(z)} - \underbrace{\frac{1}{2\pi i}\int_{|w-z_0|=r} \frac{f(w)}{(w-z)}\,dw}_{=:\, f_1(z)}$$

this is just $\frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)}\,dw = f(z)$

use uniform continuity here → $f_0(z)$ is analytic for $|z-z_0| < s$

$f_1(z)$ is analytic for $|z-z_0| > r$ and tends to 0 as $z \to \infty$

$\therefore$ We have a decomposition for $r < |z-z_0| < s$

This is independent of r, s because of the uniqueness part in ①. $\square$

Now, since f_0 is analytic in $|z-z_0| < \sigma$, we can expand it:

$$f_0(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k \qquad |z-z_0| < \sigma$$

← converges absolutely for $|z-z_0| < \sigma$
uniformly for $|z-z_0| \le r < \sigma$ any $r < \sigma$

Also,

$$f_1(z) = \sum_{k=-\infty}^{-1} a_k (z-z_0)^k \qquad |z-z_0| > \rho$$

← converges absolutely for $|z-z_0| > \rho$
uniformly for $|z-z_0| \geq r$ for $r > \rho$

<u>Defn</u>

$\therefore$ $f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$ is called the Laurent series expansion of f with respect to the annulus $\rho < |z-z_0| < \sigma$

Note that we can find the a_k by integrating around $|z-z_0| = r$ $\rho < r < \sigma$:

$$\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \int \frac{1}{(z-z_0)^{n+1}} \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k\,dz \underset{\substack{\uparrow \\ \text{uniform convergence}}}{=} \sum_{k=-\infty}^{\infty} a_k \underbrace{\int_{|z-z_0|=r} (z-z_0)^{k-n-1}\,dz}_{\begin{cases} 2\pi i & k-n-1=-1 \\ 0 & \text{otherwise}\end{cases}} = 2\pi i\, a_n$$

$$\therefore\ a_n = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad n \in \mathbb{Z}.$$

Thm (Laurent series expansion) $\quad 0 \le \rho < \sigma \le \infty$

$f(z)$ analytic for $\rho < |z - z_0| < \sigma$

Then $f(z)$ has a Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \qquad \rho < |z - z_0| < \sigma$$

which converges absolutely for $\rho < |z - z_0| < \sigma$

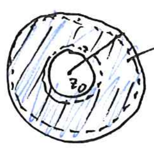
uniformly for $r \le |z - z_0| \le s \quad \rho < r < s < \sigma$.

The coefficients are given by

$$a_n = \frac{1}{2\pi i} \int_{|z - z_0| = r} \frac{f(w)}{(w - z_0)^{n+1}} dw \qquad -\infty < n < \infty$$

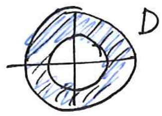
(for any $r \in (\rho, \sigma)$)

$f : D \to \mathbb{C}$ $\quad D = \{ \rho < |z - z_0| < \sigma \}$



$\sum_{-\infty}^{\infty} a_k (z - z_0)^k$ converges uniformly here

ex) $f(z) = \dfrac{1}{(z-1)(z-2)}$ on $D = \{ 1 < |z| < 2 \}$

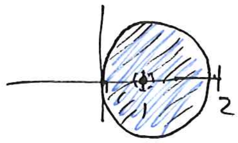


$$f_0 = \frac{1}{z-2} = -\frac{1}{2} \frac{1}{1 - z/2} = -\frac{1}{2} \left(1 + \frac{z}{2} + \frac{z^2}{4} + \dots \right)$$

$$f_1 = \frac{-1}{z-1} = -\frac{1}{z} \frac{1}{1 - \frac{1}{z}} = \frac{-1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \dots$$

$$\therefore \quad f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \qquad \begin{array}{l} a_k = -1 \text{ for } k < 0 \\ a_k = -\dfrac{1}{2^{k+1}} \text{ for } k \geq 0. \end{array}$$

ex) $f(z) = \dfrac{1}{(z-1)(z-2)}$ on $D = \{ 0 < |z-1| < 1 \}$



$$\frac{1}{z-2} = -\frac{1}{1 - (z-1)} = -\sum_{k=0}^{\infty} (z-1)^k \qquad |z-1| < 1$$

$$\rightsquigarrow \quad \frac{1}{(z-1)(z-2)} = -\sum_{k=0}^{\infty} (z-1)^{k-1}$$

$$= -\frac{1}{z-1} - 1 - (z-1) - (z-1)^2 - \dots \qquad 0 < |z-1| < 1$$

ex) Compute the Laurent series decomposition of

$$f(z) = \frac{1}{\sin z} \quad \text{in} \quad D = \{0 < |z| < \pi\}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \quad \rightsquigarrow \quad \frac{1}{\sin z} = \frac{1}{z\left(1 - \frac{1}{3!}z^2 + \frac{z^4}{5!} - \dots\right)}$$

$$= \frac{1}{z}\left(1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots\right) + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots\right)\right)$$

$$= \frac{1}{z} + \frac{z}{3!} + \frac{7z^3}{360} + \dots$$

$$\therefore f_1(z) = \frac{1}{z}$$

$$f_0(z) = \frac{z}{6} + \frac{7z^3}{360} + \dots$$

ex) $f(z) = \dfrac{z^2 - \pi^2}{\sin z} = \underbrace{\sum_{k=-\infty}^{\infty} a_k (z-0)^k}_{\text{Laurent expansion}}$ $\quad z_0 = 0$

Q: What is the largest open set s.t $\uparrow$ which converges for $|z| = 1$. converges?

$$f(z) = \frac{(z-\pi)(z+\pi)}{\sin z}$$

$\sin z$ has a simple zero at $z = \pi$ $\rightsquigarrow$ f extends to an analytic function at $z = \pi$

—— " —— at $z = -\pi$

$\rightsquigarrow$ f converges on $\{0 < |z| < 2\pi\}$ $\leftarrow$ cannot make this larger because of the zero at $z = 2\pi$.

Isolated singularities

Defn A function f has an isolated singularity at $z_0 \in \mathbb{C}$ if $\exists\, r > 0$ s.t. $f(z)$ is analytic on $\{0 < |z - z_0| < r\}$ ← punctured disk

ex1 $f(z) = \frac{1}{z^n}$ isolated singularity at $z = 0$

$f(z) = \frac{1}{\sin(\pi z)}$ —— 〃 —— at $z = 0, \pm 1, \pm 2$

$f(z) = \sqrt{z}$ does not have any isolated singularities at $z = 0$ (any disk must intersect the negative real axis)

If f has an isolated singularity at z_0 then there is a Laurent expansion of the form

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \qquad 0 < |z - z_0| < r$$

Defn
- z_0 is a removable singularity if $a_k = 0 \;\; \forall\, k < 0$
- z_0 is a pole of order N if $a_{-N} \neq 0$ but $a_j = 0 \;\; \forall\, j < -N$
- z_0 is an essential singularity if $a_k \neq 0$ for infinitely many $k < 0$.

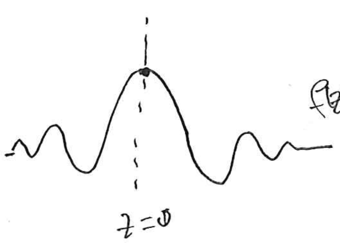
Standard example of a removable singularity:

$f(z) = \frac{\sin z}{z}$ $\quad z_0 = 0$: $\quad f(z) = \frac{\sin z}{z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$

⇝ Can extend f ("remove the singularity") to $z = 0$, by defining $f(0) = 1$.

Important theorem:

Thm (Riemann's theorem on removable singularities) Let z_0 be an isolated singularity of $f(z)$. If f is bounded near z_0, then z_0 is a removable singularity.



$f(z) = \frac{\sin z}{z}$

$z = 0$

note: $|f(z)| \leq 1$ for z near 0.

Proof. Suppose $|f(z)| \leq M$ for all $|z-z_0| < \rho$

Let $r << \rho$ (be small)

~~From~~ z_0 isolated $\Rightarrow$ $f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$ $\quad |z-z_0| < r << \rho$

$$\therefore \quad |a_n| \leq \frac{1}{2\pi} \frac{M}{r^{n+1}} (2\pi r) = \frac{M}{r^n}$$

$n < 0 \Rightarrow$ RHS tends to 0 as $r \to 0 \Rightarrow a_n = 0 \ \forall n < 0 \Rightarrow z_0$ removable. $\square$

~~Defn~~ An isolated singularity z_0 is a pole (of order N). if $a_{-N} \neq 0$ but $a_k = 0$ $k < -N$

The Laurent expansion of f has the form

$$\therefore \ f(z) = \underbrace{\frac{a_{-N}}{(z-z_0)^N} + \frac{a_{-N+1}}{(z-z_0)^{N-1}} + \cdots + \frac{a_{-1}}{z-z_0}}_{=: \text{ the principal part of } f \ \ P(z)} + a_0 + a_1 z + \ldots$$

ex) $f(z) = \dfrac{\sin(z+i)}{(z-2)^3}$ has a pole of order 3 at $z=2$.

Note $P(z) = f_0(z)$.

$f(z) - P(z)$ is analytic at z_0 $\rightarrow$ N measures how "bad" the singularity of $f(z)$ is at $z = z_0$.

$N=1 \rightsquigarrow$ "simple pole" $\quad \left(f(z) = \frac{1}{z} \right)$

$N=2 \rightsquigarrow$ "double pole". $\quad \left(f(z) = \frac{1}{(z-i)^2} \right)$

Thm z_0 isolated singularity of f

~~z_0 ...~~ z_0 is a pole of order $N \iff f(z) = \dfrac{g(z)}{(z-z_0)^N}$ where $g(z_0) \neq 0$ (g analytic)

proof ($\Rightarrow$) f has a pole of order N at $z = z_0$:

$\Rightarrow a_{-N} + a_{-N+1}(z-a_0) + \cdots +$ converges to an analytic function $g(z)$

and $f(z) = \dfrac{g(z)}{(z-z_0)^N}$ (also $g(z_0) \neq 0$).

($\Leftarrow$): $f(z) = \dfrac{1}{(z-z_0)^N} \left(\sum_{k=0}^{\infty} b_k (z-z_0)^k \right) \Rightarrow f(z) = \dfrac{b_0}{(z-z_0)^N} + \dfrac{b_1}{(z-z_0)^{N-1}} + \cdots$

and $b_0 \neq 0$

$\Rightarrow f$ has a pole of order N at $z = z_0$

$\square$

Thm z_0 isolated singularity of $f(z)$.

z_0 is a pole of order $N \iff \frac{1}{f(z)}$ analytic at $z = z_0$ and $\mathrm{ord}_{z_0}\left(\frac{1}{f(z)}\right) = N$.

proof f pole of order $N \iff f(z) = (z-z_0)^{-N} g(z) \quad g(z_0) \neq 0$

$h(z) = \frac{1}{g(z)} = \frac{1}{(z-z_0)^N f(z)}$ is analytic at $z = z_0$ and $h(z_0) \neq 0$

$\iff \frac{1}{f(z)} = \frac{1}{h(z)} \cdot (z-z_0)^N$ is analytic at $z = z_0$ and $\mathrm{ord}_{z_0}\left(\frac{1}{f}\right) = N$. $\square$

ex) $f(z) = \frac{1}{\sin z}$ has simple poles at $z = 0, \pm 1, \pm 2, \ldots$

(since $\mathrm{ord}_{z=n\pi}(\sin z) = 1$).

We can extend $\mathrm{ord}_a(f)$ for a an isolated singularity of f:

$$\mathrm{ord}_a(f) = \left\{ n \,\middle|\, \begin{array}{l} f(z) \text{ can be written as } (z-a)^n g(z) \\ \text{where } g \text{ is analytic and } g(a) \neq 0 \end{array} \right\}$$

$$= \begin{cases} \mathrm{ord}_a(f) & \text{if } f \text{ is analytic at } z = a \\ -\mathrm{ord}_a\left(\frac{1}{f}\right) & \text{if } f \text{ has a pole at } z = a \end{cases}$$

ex)

$\therefore \mathrm{ord}_a\left(\frac{1}{z-a}\right) = -1$

ex) More generally, $\mathrm{ord}_a\left(\frac{f(z)}{g(z)}\right) = \mathrm{ord}_a(f(z)) - \mathrm{ord}_a(g(z))$

$\mathrm{ord}_1\left(\frac{z^3-1}{z^2-1}\right) = \mathrm{ord}_1(z^3-1) - \mathrm{ord}_1(z^2-1) = 1 - 1 = 0$

$= \mathrm{ord}_1\left(\frac{z^2+z+1}{z+1}\right) = 0 \quad \left(\frac{z^2+z+1}{z+1}\right.$ is analytic at $\left.z=1\right)$

ex) $f(z) = \frac{z^2-\pi^2}{\sin(z)}$

$\mathrm{ord}_\pi f = \mathrm{ord}_\pi(z^2-\pi^2) - \mathrm{ord}_\pi(\sin z) = 1 - 1 = 0 \quad \therefore \frac{z^2-\pi^2}{\sin z}$ has a removable singularity at $z = \pi$

$z^2 - \pi^2 = (z-\pi)(z+\pi)$

Cor z_0 isolated singularity of $f(z)$. Then z_0 is removable if ~~and~~ only if $\operatorname{ord}_{z_0}(f) =$

Extended example $f(z) = \frac{1}{\sin z}$ $\quad D = \{ \pi < |z| < 2\pi \}$ ← largest annulus containing $|z|=4$ where f is analytic.

$$\frac{1}{\sin z} = \frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots} = \frac{1}{z} \frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots} = \frac{1}{z}\left(1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots\right) + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots\right)^2 + \cdots\right)$$

$$= \frac{1}{z}\left(1 + \frac{z^2}{3!} - \frac{7z^4}{360} + \cdots\right)$$

$\therefore \frac{1}{\sin z} = \frac{1}{z} +$ analytic

also $\frac{1}{\sin z} = \frac{1}{z-\pi} +$ analytic $\quad$ also $\frac{1}{\sin z} + \frac{1}{z+\pi}$ analytic

Define $f_1(z) = \frac{1}{z} - \frac{1}{z+\pi} - \frac{1}{z-\pi}$

$\Rightarrow$ $f_0(z) = \frac{1}{\sin z} - f_1(z)$ is analytic for $|z| < 2\pi$.

$\therefore$ $f_0(z) + f_1(z)$ is the Laurent decomposition of f ~~at~~ in D.

Note: $f_1(z) = \frac{1}{z} - \frac{1}{z+\pi} - \frac{1}{z-\pi} = \frac{1}{z} - \frac{2z}{z^2 - \pi^2} = \frac{1}{z} - \frac{2}{z}\sum_{k=0}^{\infty} \frac{\pi^{2k}}{z^{2k}} = -\frac{1}{z} - \sum_{k=1}^{\infty} \frac{2\pi^{2k}}{z^{2k+1}}$

The terms of $f_0(z)$ are harder to compute. $\quad = -\frac{1}{z} - \frac{2\pi^2}{z^3} + \cdots$

Defn A function $f: D \to \mathbb{C}$ is <u>meromorphic</u> if $f(z)$ is analytic on D with an exception of a set of isolated set of singularities, which are poles (in D).

$M(D)$ = set of meromorphic functions $f: D \to \mathbb{C}$

Note: $f, g \in M(D) \Rightarrow$ $f+g \in M(D)$ and $f \cdot g \in M(D)$

$f-g \in M(D)$

$f \cdot g \in M(D)$

$f/g \in M(D)$ (if $g \not\equiv 0$)

← $M(D)$ is a field.

ex| rational functions $f(z) = \frac{P(z)}{Q(z)}$ are meromorphic:

$$= \frac{c(z-a_1)^{m_1} \cdots (z-a_k)^{m_k}}{(z-b_1)^{n_1} \cdots (z-b_\ell)^{n_\ell}} \qquad c \neq 0$$

- $\mathrm{ord}_{a_i}(f) = m_i$ ← zero of order m_i
- $\mathrm{ord}_{b_i}(f) = -n_i$ ← pole of order m_i.

ex| $\frac{1}{\sin z}$ is meromorphic (with poles at $z = n\pi$ $n \in \mathbb{Z}$) ~~zeroes at~~ (but not rational).

note: $e^{1/z}$ does not lead to ∞ if $z \to 0$: If $z = ri$ r real then $e^{1/ri} = e^{-i \cdot \frac{1}{r}}$ which has absolute value 1..

Thm z_0 an isolated singularity of $f(z)$.
z_0 is a pole $\Leftrightarrow$ $|f(z)| \to \infty$ as $z \to z_0$.

proof. ($\Rightarrow$): If z_0 is a pole of order N $\Rightarrow$ $g(z) = (z-z_0)^N f(z)$ is analytic and $g(z_0) \neq 0$

$\Rightarrow |f(z)| = |z-z_0|^{-N} |g(z)| \longrightarrow \infty$.

($\Leftarrow$): If $|f(z)| \to \infty$ as $z \to z_0$, then $f(z) \neq 0$ for z in a disk D near z_0.

$\Rightarrow h(z) = \frac{1}{f(z)}$ analytic in some $0 < |z-z_0| < r$.
and $h(z) \to 0$ as $z \to z_0$

$\Rightarrow$ h extends to an analytic function at z_0 (and $h(z_0) = 0$)
Riemann's theorem

If $\mathrm{ord}_{z_0} h = N$ $\Rightarrow$ $\mathrm{ord}_{z_0}(f(z)) = -N \Rightarrow f$ has a pole of order N at z_0 □

Defn z_0 is an essential singularity if $a_k \neq 0$ for infinitely many $k < 0$ in the Laurent expansion of f at z_0.

Conclusion
∴ We have 3-types of singularities:

- removable singularity: $\frac{\sin z}{z}$
- pole of order N $\frac{1}{z^N}$
- essential singularity: $e^{\frac{1}{z}}$

prototype of an essential singularity:

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!} \cdot \frac{1}{z^2} + \frac{1}{3!} \frac{1}{z^3} + \ldots$$

note that it is indeed essential!

Thm (Casorati–Weierstrass theorem) Let z_0 be an essential singularity of f. Then for any $w_0 \in \mathbb{C}$, there is a sequence $z_n \to z_0$ such that

$$f(z_n) \to w_0$$

Pretty strange: the values of f near z_0 "cluster" towards the whole complex plane
this is not true for $f(z) = \frac{1}{z}$ for instance.

Proof. Suppose the theorem is false.
Then $\exists w_0$ which is not the limit of $f(z_n)$ for a sequence (z_n).

$\Rightarrow \exists \varepsilon > 0$ s.t $|f(z) - w_0| > \varepsilon \quad \forall z$ near z_0

$\Rightarrow h(z) = \dfrac{1}{f(z) - w_0}$ bounded near z_0

$\Rightarrow h(z)$ has a removable singularity at z_0

$\Rightarrow h(z) = (z - z_0)^N g(z)$ for some N

$\Rightarrow f(z) = w_0 + (z - z_0)^{-N} \underbrace{\left(\dfrac{1}{g(z)}\right)}_{\text{analytic at } z_0}$ $\leftarrow$ if $N = 0$ we have a removable singularity
$N > 0$ we have a pole

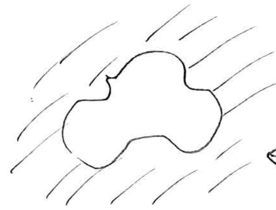
$\Rightarrow$ contradiction $\Rightarrow$ proof complete $\square$

Isolated singularities at ∞

$f(z)$ has an isolated singularity at $z=\infty$ if $g(w)=f\left(\frac{1}{w}\right)$ has an isolated singularity at $w=0$.

$\leftarrow$ This is $\Leftrightarrow$ $f(z)$ is analytic outside some bounded set

Also, $z=\infty$ is a removable singularity; pole; essential singularity if $w=0$ is for $g(w)$.



f analytic here

Let $f(z)=\sum_{k=-\infty}^{\infty} b_k z^k$ be a Laurent expansion in $|z|>R$.

$\therefore$ $z=\infty$ is removable if $b_k=0$ $\forall k>0$ (ex $f(z)=\frac{1}{z}$)

pole of order N if $b_k=0$ $\forall k>N$ $(b_N\neq 0)$ (ex polynomials of degree N

essential if $b_k\neq 0$ for infinitely many $k>0$. (ex $e^z=1+z+\frac{z^2}{2!}+\ldots$

If $f(z)$ has a pole of order N at $z=\infty$ $\Rightarrow$ $f(z)=\underbrace{b_N z^N+b_{N-1}z^{N-1}+\ldots+b_0}_{=:P(z),\text{ the principal part of } f(z)}+\frac{b_1}{z}+\ldots$

$\therefore$ $f(z)-P(z)$ is analytic at $z=\infty$ (and is zero there).

ex $e^z=1+z+\frac{z^2}{2!}+\ldots$ essential singularity at $z=\infty$

$D \subset \mathbb{C}^*$ (domain in the extended complex plane)

Defn $\quad f: D \to \mathbb{C}^*$ is <u>meromorphic</u> if f is analytic on D except for a set of isolated singularities each of which is a pole.

$$\mathcal{M}(D) = \{ f: D \to \mathbb{C}^* \mid f \text{ meromorphic} \}$$

$\mathcal{M}(D)$ is a field: $\quad f, g \in \mathcal{M}(D) \Rightarrow \quad f \pm g \in \mathcal{M}(D)$

$fg \in \mathcal{M}(D)$

$\frac{f}{g} \in \mathcal{M}(D) \quad (\text{if } g \neq 0)$

ex $\quad f(z) = \dfrac{z^3 - 3}{z - 1}$ is meromorphic on $D = \mathbb{C}^*$

ex $\quad f(z) = e^z = 1 + \cancel{z} + \frac{z^2}{2!} + \dots$ is meromorphic on $D = \mathbb{C}$, but not on $D = \mathbb{C}^*$: It has an <u>essential</u> singularity at $z = \infty$.

ex $\quad f(z) = \dfrac{p(z)}{q(z)}$ rational function $\Rightarrow f$ is meromorphic: finite number of poles $(q(z) = 0)$

a finite pole at $z = \infty$:

$f(z) = d(z) + \dfrac{r(z)}{q(z)}$ $\quad d(z)$ a polynomial

$\dfrac{r(z)}{q(z)} \to 0 \quad z \to \infty$

ex $\quad f(z) = \dfrac{e^{1/z} - 1}{z} = \left(\frac{1}{z} + \frac{1}{2!} \frac{1}{z^2} + \frac{1}{3!} \frac{1}{z^3} + \dots \right.$

$\therefore$ The singularity at $z = \infty$ is removable $\rightsquigarrow$ can define f analytically on $D = \mathbb{C}^*$ by defining $f(\infty) = 0$.

Thm A meromorphic function on $\mathbb{C}^*$ is rational

$$\therefore \mathcal{M}(\mathbb{C}^*) = \left\{ \frac{p(z)}{q(z)} \;\middle|\; p, q \text{ polynomials } (q \not\equiv 0) \right\}$$

Proof Let $f \in \mathcal{M}(\mathbb{C}^*)$

$\Rightarrow$ f has finitely many poles (otherwise they would accumulate somewhere on $\mathbb{C}^*$ {which is compact} and the singularity would not be isolated).

If f is analytic at ∞, define $P_\infty(z) = f(\infty)$.
Otherwise, f has a pole of order N at $z=\infty$

$$P_\infty(z) = \text{principal part of } f \text{ at } z=\infty = b_N z^N + b_{N-1} z^{N-1} + \dots + b_0$$

$$\Rightarrow \underbrace{f(z) - P_\infty(z)}_{\substack{\text{analytic at} \\ z=\infty.}} \xrightarrow[z\to\infty]{} 0$$

Let $z_1 \dots z_m$ be the set of poles of f in $\mathbb{C}$.

$$P_k(z) = \text{principal part of } f \text{ at } z_k = \frac{a_1}{z-z_k} + \frac{a_2}{(z-z_k)^2} + \dots + \frac{a_n}{(z-z_k)^{n_k}}$$

$$P_k(z) \xrightarrow[z\to\infty]{} 0$$

$\leftarrow$ analytic at $z=\infty$.

$$\therefore \text{ Define } \quad g(z) = f(z) - P_\infty(z) - \sum_{k=1}^{m} P_k(z)$$

g analytic at $z_1 \dots z_m$ and $z=\infty$ ✓

$\Rightarrow$ $g(z)$ an entire function

$\Rightarrow$ (Liouville) $g(z) = \text{constant} \Rightarrow g(z) \equiv 0$ (since $g(\infty) = 0$)

$$\Rightarrow f(z) = P_\infty(z) + \sum_{k=1}^{m} P_k(z) = \text{rational function in } z \qquad \square$$

Thm Every rational function $f \in M(\mathbb{C}^*)$ has a partial fractions decomposition

$$f(z) = P_\infty(z) + \sum_{k=1}^{m} P_k(z)$$

where

(i) $P_\infty(z)$ is a polynomial

(ii) $P_k(z) = \frac{a_1}{(z-z_k)} + \frac{a_2}{(z-z_k)^2} + \dots + \frac{a_{n_k}}{(z-z_k)^{n_k}}$ principal part around $z = z_k$.

This really is just the familiar "partial fractions decompositions" you know from MAT1100:

ex) $f(z) = \frac{z^3+z+1}{z^2+1}$

$$= z + \underbrace{\frac{1}{z^2+1}}_{\text{poles at } z = \pm i} \qquad \frac{1}{z^2+1} = \frac{A}{z+i} + \frac{B}{z-i}$$

$\leadsto$ $A(z-i) + B(z+i) = 1$ $\quad z = -i \quad 1 = -2i\,A$

$z = i \quad 1 = 2i\,B$

$\leadsto$ $f(z) = z + \frac{1}{2i}\frac{1}{z-i} - \frac{1}{2i}\frac{1}{z+i}$

$P_\infty \qquad P_1 \qquad P_2$

ex) $f(z) = \frac{z+1}{z^4 - 3z^3 + 3z^2 - z} = \frac{z+1}{z(z-1)^3}$

f has a pole of order 1 at $z=0$

— " — 3 at $z=1$ $\Rightarrow$ $f(z) = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{(z-1)^2} + \frac{D}{(z-1)^3}$

$$1+z = A(z^3 - 3z^2 + 3z - 1) + B(z^3 - 2z^2 + z) + C(z^2 - z) + Dz \qquad (*)$$

$\frac{\partial}{\partial z}$: $\quad 1 = A(3z^2 - 6z + 3) + B(3z^2 - 4z + 1) + C(2z-1) + D \qquad (**)$

$\frac{\partial}{\partial z}$: $\quad 0 = A(6z-6) + B(6z-4) + 2Cz \qquad (***)$

$\frac{\partial}{\partial z}$: $\quad 0 = 6A + 6B \qquad (****)$

$\therefore \quad A = -B$

Set $z = 1$ in $(*)$: $\quad 1 + 1 = 0 + D \Rightarrow D = 2.$

Set $z = 1$ in $(**)$: $\quad 1 = C(\cancel{z}-1) + 2 \quad \therefore \quad C = -1$

Set $z = 1$ in $(***)$: $\quad 0 = 2B + 2$

$$\therefore \quad \frac{z+1}{z^{4} - 3z^{3} + 3z^{2} - z} = -\frac{1}{z} + \frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{2}{(z-1)^3}$$

What is the Laurent series in $D = \{0 < |z-1| < 1\}$?

$$\frac{-1}{z} = \frac{-1}{1 + (z-1)} = \sum_{k=0}^{\infty} (-1)^{k+1} (z-1)^k$$

$$\therefore \quad f(z) = \frac{2}{(z-1)^3} - \frac{1}{(z-1)^2} + \frac{1}{(z-1)} + \sum_{k=0}^{\infty} (-1)^{k+1} (z-1)^k$$

is the Laurent series about $z_0 = 1$.

At $z_0 = 1$:

$$\frac{1}{z-1} = -\sum_{k=0}^{\infty} z^k$$

$$\frac{1}{(z-1)^2} = \sum_{k=0}^{\infty} (k+1) z^k \longrightarrow f(z) = -\frac{1}{z} + \frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{2}{(z-1)^3}$$

$$\frac{1}{(z-1)^3} = -\sum_{k=0}^{\infty} \frac{(k+2)(k+1)}{2} z^k$$

$$= -\frac{1}{z} + \sum_{k=0}^{\infty} \left(1 + (k+1) + (k+2)(k+1)\right) z^k$$

$$= -\frac{1}{z} + \sum_{k=0}^{\infty} (k^2 + 4k + 4) z^k$$

$$= -\frac{1}{z} + \sum_{k=0}^{\infty} (k+2)^2 z^k.$$

$$= -\frac{1}{z} + 4 + 9z + 16z^2 + \dots$$

H12

problem 2

Find the Laurent series for $f(z) = \dfrac{1}{4z - z^3}$

in a) $D = \{ 0 < |z| < 2 \}$

and b) $D = \{ |z| > 2 \}$

) Classify the singularities of $f(z)$

a) $\dfrac{1}{z} \dfrac{1}{4 - z^2} = \dfrac{1}{4z} \dfrac{1}{\underbrace{1 - \frac{z^2}{4}}_{|\frac{z}{2}|^2 < 1}} = \sum_{n=0}^{\infty} \dfrac{1}{4^{n+1}} z^{2n-1}$

) $\dfrac{1}{z} \dfrac{1}{4 - z^2} = \dfrac{1}{z}\left(-\dfrac{1}{z^2} \dfrac{1}{1 - 4/z^2} \right) = -\sum_{n=0}^{\infty} 4^n z^{-2n-3} = -z^{-3} - 4z^{-5} - 16z^{-7} - \dots$

) $P = 4z - z^3 = z(4 - z^2) = z(2-z)(2+z)$

~~then~~ $\operatorname{ord}_a f(z) = -\operatorname{ord} P(z) = \begin{cases} -1 & \text{if } z = 0, 2, -2 \\ 0 & \text{otherwise} \end{cases}$

∴ All singularities are poles of order 1

Problem 3

$$f(z) = \frac{z^2+1}{z^3+i}$$

determine the singularities and determine their type.

$$\begin{array}{l} z^3 + \\ \underline{z^3 - iz^2} \\ iz^2 \\ \underline{iz^2 + z} \\ -z - i \end{array}$$

$i : z - i = z^2 + iz -$

$z^3 + i \overset{?}{=} (z-i)(z^2+iz-1)$

Solution: $z = i$ is a zero of both z^2+1 and z^3+i :

$$\operatorname{ord}_i f(z) = \operatorname{ord}_i (z^2+1) - \operatorname{ord}_i (z^3+i)$$
$$= \operatorname{ord}_i (z+i) + \operatorname{ord}_i (z-i) - \operatorname{ord}_i (z-i) - \operatorname{ord}_i (z^2+iz-1)$$
$$= 0 + 1 - 1 - 0 = 0$$

$\rightarrow$ i is a removable singularity by Riemann's theorem

$z^2 + iz - 1 = 0 \quad \rightsquigarrow \quad z = \frac{-i \pm \sqrt{i^2 - 4\cdot(-1)}}{2} = \frac{-i \pm \sqrt{3}}{2}$

$\rightsquigarrow$ simple poles.

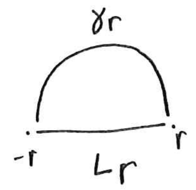
Compulsory assignment 2 #12

$f(z) = \frac{1}{1+z^2}$

$r > 0$

$L_r = [-r, r]$

γ_r = upper half circle with radius r



1) Compute $\int_{L_r} f(z)\,dz$

2) Use Cauchy's integral formula to compute $\int_{\gamma_r} f(z)\,dz$ $r < 1$

3) Compute $\int_{\gamma_r} f(z)\,dz$ when $r > 1$

4) Find the Taylor series for $f(z)$ near 0 and determine radius convergence

5) Find the Laurent series for

$$g(z) = \frac{1}{z(1+z^2)} \quad \text{in } D = \{0 < |z| < 1\}$$

a) $$\int_{-r}^{r} \frac{dz}{1+z^2} = \arctan z \Big|_{z=-r}^{r}$$

$$= \arctan r - \arctan(-r) = 2\arctan r$$

b) Let $D =$ 

$f(z)dz$

Cauchy: $\int_{\partial D} f(z)\, dz = 0$ (since f is analytic) $\boxed{r<1}$

$$\therefore \int_{\gamma_2} f(z) dz = \int_{\partial D} f\, dz - \int_{L_\gamma} f(z) = -2\tan^{-1}(r).$$

c) $r>1$:

$$\frac{1}{z^2+1} = \frac{A}{z+i} + \frac{B}{z-i} \quad \therefore \quad A(z-i) + B(z+i) = 1 \qquad z=i$$

$$= -\frac{1}{2i}\frac{1}{z+i} + \frac{1}{2i}\frac{1}{z-i} \qquad 2i\cdot B = 1 \qquad z=-i$$

$$-2i\, A = 1$$

$$\therefore \int_{\Delta} f(z)\, dz = \int_{\Delta} \frac{1}{2i}\frac{dz}{z-i}$$

since $\frac{1}{z+i}$ analytic in Δ

$$= \frac{1}{2i}\int_{\Delta'} \frac{dw}{w} \quad \text{where } w = z-i \qquad \frac{1}{z-i} =$$

$$\Delta' = \Delta - i$$

$$= \frac{1}{2i} \cdot 2\pi i = \underline{\pi}.$$

Alternatively, by Cauchy

$h(z) = \frac{1}{z+i}$

$$\int_{\Delta} f(z)\, dz = \int \frac{h(z)}{z-i} dz$$

$$= h(i)\cdot 2\pi i$$

$$= \frac{1}{2i} 2\pi i = \pi.$$

$$\therefore \int_{\gamma_r} f(z)\, dz = \pi - \arctan r$$

d) $$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n} = 1 - z^2 + z^4 - \dots$$

has radius of convergence $R = 1$ = distance from 0 to nearest singularity (i)

The Laurent series of

$$g(z) = \frac{1}{z(1+z^2)}$$

is given by

$$\frac{1}{z}\,\frac{1}{1+z^2} = \frac{1}{z}\left(1 - z^2 + z^4 - \cdots\right)$$

$$= z^{-1} - z + z^3 - \cdots$$