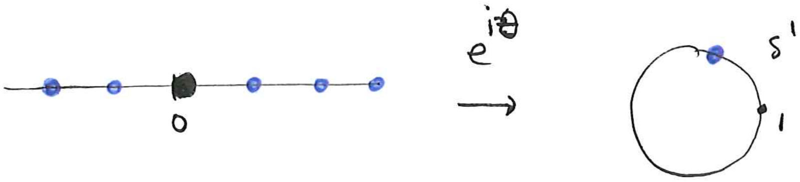
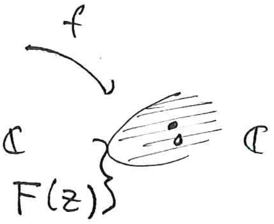
Periodic functions and Fourier series

$S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$ ← unit circle

What are the functions on S^1?





Functions $f : S^1 \to \mathbb{C}\}$ $\longleftrightarrow$ $\{$Functions $F : \mathbb{R} \to \mathbb{C}$ s.t. $F(z+2\pi) = F(z)\}$
(the 2π-periodic functions)

Defn $w \in \mathbb{C}$. D domain
$f : D \to \mathbb{C}$ is called <u>w-periodic</u> (or just <u>periodic</u>) if

$$f(z+w) = f(z) \qquad \forall z \in D$$

← w is called a <u>period</u> of f.

Ex e^z is $2\pi i$-periodic (all periods: $0, \pm 2\pi i, \pm 4\pi i, \pm \ldots$)
e^{iz} is 2π-periodic (all periodic: $0, \pm 2\pi, \pm 4\pi, \ldots$)
$\sin z$ and $\cos z$ are periodic with period 2π:

$$\sin z = \tfrac{1}{2} e^{iz} - \tfrac{1}{2} e^{-iz}$$

$$\cos z = \tfrac{1}{2} e^{iz} + \tfrac{1}{2} e^{-iz}$$

← we will show that any 2π-periodic function is a sum of exponentials..

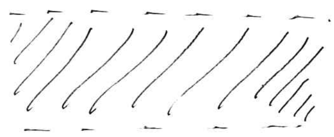
Note: If $w \neq 0$ is a period of f
$\Rightarrow g(z) = f(w \cdot z)$ is periodic with period 1:

$$g(z+1) = f(w(z+1)) = f(wz+w) = f(wz) = g(z).$$

When studying periodic functions, we can assume wlog $w = 1$.

Primary examples: $e^{2\pi i k z}$ $\quad k \in \mathbb{Z}$.

Thm Let $D = \{ z \mid \alpha < \operatorname{Im} z < \beta \}$



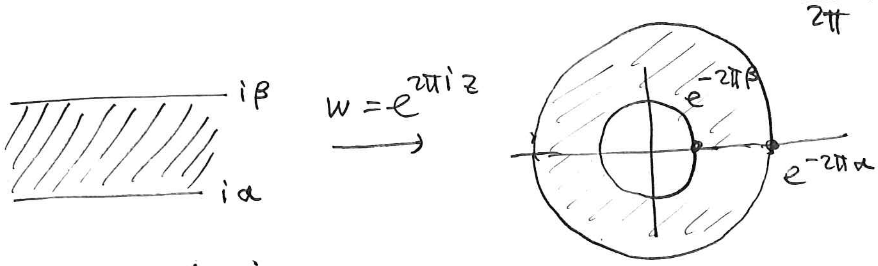
If $f : D \to \mathbb{C}$ is analytic and periodic with period 1, then $f(z)$ can be expanded as a series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi i k z}$$

Absolutely convergent for all $z \in D$

Uniformly convergent for all $z \in \{ \alpha_0 \le \operatorname{Im} z \le \beta_0 \}$ $\quad \alpha < \alpha_0 < \beta_0 < \beta$

proof Let $w = e^{2\pi i z} \leadsto z = -\frac{i}{2\pi} \operatorname{Log} |w| + \frac{\arg w}{2\pi}$



Let $g(w) = f\left(-\frac{i}{2\pi} \operatorname{Log} |w| + \frac{\operatorname{Arg} w}{2\pi} \right) = f(z)$

f periodic with period 1 $\Rightarrow$ g is well-defined (independent of choice of branch of logarithm)

$\Rightarrow$ g analytic in $e^{-2\pi\beta} < |w| < e^{-2\pi\alpha}$

$\Rightarrow$ g has Laurent expansion $g(w) = \sum_{k=-\infty}^{\infty} a_k w^k$

$\Rightarrow$ $f(z) = g(w) = \sum_{k=-\infty}^{\infty} a_k w^k = \sum_{k=-\infty}^{\infty} a_k e^{2\pi i k z}$.

Thm Let $D = \{ z \in \mathbb{C} \mid \operatorname{Im} z > \alpha \}$:

$: D \to \mathbb{C}$ analytic + periodic with period 1.
If f is bonded as $\operatorname{Im} z \to \infty$, then f has a series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k e^{2\pi i k z} \qquad \operatorname{Im} z > \alpha.$$

This is uniformly convergent on $\{ \operatorname{Im} z \geq \alpha_0 \}$ $\quad \alpha_0 > \alpha$.

Proof. $g(w) = f(z)$ as before.

f bonded as $\operatorname{Im} z \to \infty$ $\Rightarrow$ g bonded as $w \to 0$

Riemann's theorem $\Rightarrow$ g extends to an analytic function at $w = 0$

$$\Rightarrow g(w) = \sum_{k=0}^{\infty} a_k w^k \qquad |w| < e^{-2\pi\alpha}$$

$$\Rightarrow f(z) = \sum_{k=0}^{\infty} a_k e^{2\pi i k z}.$$

Ex1 $f(z) = \dfrac{1}{1 - \cos(2\pi z)}$ on $D = \{ \operatorname{Im} z > 0 \}$

(note that $1 - \cos(2\pi z) = 0 \Rightarrow z = 0, \pm 1, \pm 2, \ldots$ which are not in D..)

$z = x + iy$

$$\Rightarrow |\cos z|^2 = \cos z \cdot \cos \bar{z}$$

$$= \frac{1}{4}\left(e^{iz} + e^{-iz}\right)\left(e^{i\bar{z}} + e^{-i\bar{z}}\right) = \frac{1}{4}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right)\left(e^{i(x-iy)} + e^{-i(x-iy)}\right)$$

$$= \frac{1}{4}\left(e^{-y+ix} + e^{+y-ix}\right)\left(e^{-y+ix} + e^{y-ix}\right) = \frac{1}{4}\left(e^{2ix} + e^{-2y} + e^{2y} + e^{-i2x}\right) \xrightarrow{y \to \infty} \infty$$

$\therefore f(z) \to 0$ as $\operatorname{Im} z \to \infty$

$\Rightarrow f$ has an expansion of the form $\displaystyle\sum_{k=0}^{\infty} a_k e^{2\pi i k z}$.

What is this expansion?

$$\frac{1}{1-\left(\frac{e^{2\pi i z}+e^{-2\pi i z}}{2}\right)} = -\frac{2e^{2\pi i z}}{(1+e^{2\pi i z})^2}$$

$$= -2e^{2\pi i z}\left(1-2e^{2\pi i z}+3e^{4\pi i z}-4e^{6\pi i z}+\cdots\right)$$

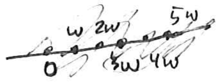
$$-\frac{2w}{(1+w)^2} = -2w(1-2w+3w^2-\cdots)$$

What about (meromorphic) functions with several periods?

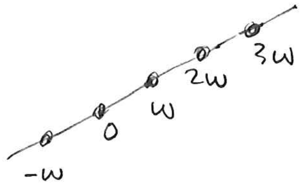
<u>Thm</u> $f(z)$ meromorphic function (non-constant) which is periodic. Then either:

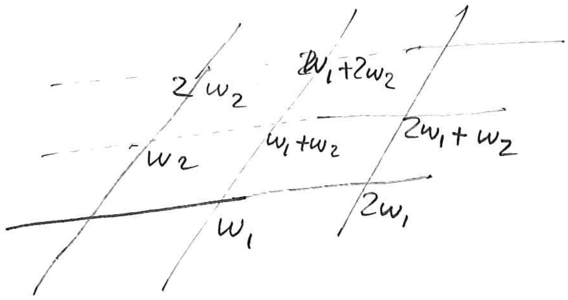
(i) There is a period ω s.t all periods are of the form $n\omega$ $n \in \mathbb{Z}$; or

(ii) There are two periods ω_1, ω_2 s.t all periods are of the form $m\omega_1 + n\omega_2$ $m, n \in \mathbb{Z}$.



Proof:





Claim: The set of periods is ~~discrete~~ isolated (any bounded set contains only finitely many periods)

Let $a \in D$. $g(z) = f(z+a) - f(a)$ has an isolated zero at $z=0$

$\Rightarrow$ $\exists\, \rho > 0$ s.t $g(z) \neq 0$ $\forall\, 0<|z|<\rho$.

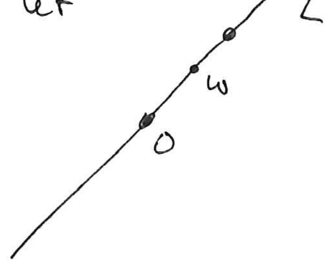
$\Rightarrow$ $|\omega| \geq \rho$ for any period ω (since $g(\omega) = f(\omega+a) - f(a) = 0$)

$\Rightarrow$ $|\omega_1 - \omega_2| \geq \rho$ for any two periods ω_1, ω_2 (since $\omega_1 - \omega_2$ is also a period)

$\Rightarrow$ Every period can be separated from the others by a small disk of radius $< \rho/2$

$\Rightarrow$ the set of periods is isolated.

Let $L \subset \mathbb{C}$ be a line containing a period and let ω = period on L of smallest $|\omega|$ and $\operatorname{Arg} \omega$.



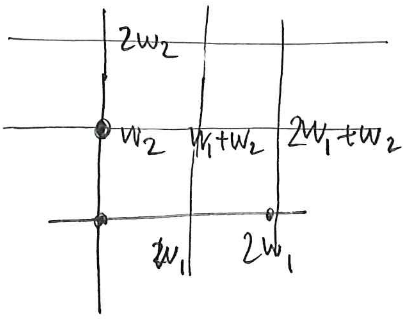
$\Rightarrow$ Any period on L has the form $n\omega$.

(if not, we can find ω' between $k\omega$ and $(k+1)\omega$

$\rightarrow$ subtracting $k\omega$ we find one between 0 and ω $\Rightarrow$ contradicting our choice of ω).

If all the periods lie on $L \Rightarrow$ ok.

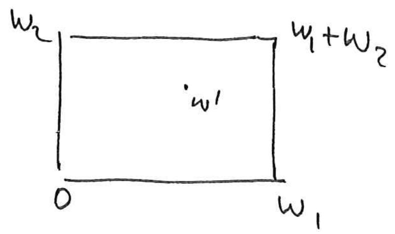
If not, $\exists\, w_2$ which is closest to $[0, w_1]$ and smallest arg w:



$\therefore$ Can use w_1 and w_2 as a $\mathbb{R}$-basis for $\mathbb{C}$.

<u>Claim</u>: All periods are of the form $m w_1 + n w_2$.

This follows by our choice of w_2: If w is a period then $w = m w_1 + n w_2 + w'$ for some w' in the rectangle



However, in this case either w' or $w_1 + w_2 - w'$

is closer to $[0, w_1]$ than w_2.

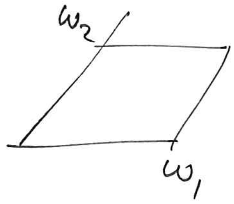
$\square$

Defn f is simply periodic if all periods lie on a line through 0
f is doubly periodic if $\exists$ two periods ω_1, ω_2 not on a line

ex e^z, $\cos z$, $\sin z$ are simply periodic (prove this!)

ex

$$f(z) = \sum_{m,n=-\infty}^{\infty} \frac{1}{(z-(m\omega_1+n\omega_2))^k} \qquad k \geq 3$$



is a doubly periodic function. f is meromorphic with poles at $z = m\omega_1 + n\omega_2$ $\forall\, m,n \in \mathbb{Z}$.

checking this requires a computation

Fourier Series

Defn A complex Fourier series is a series of the form

$$F(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt} \qquad (*)$$

Note: If $f(z)$ has a Laurent expansion on $|z|=1$ $\sum_{k=\infty}^{\infty} a_k z^k$, then $F(t) = f(e^{it})$ has a Fourier expansion $\sum_{k=\infty}^{\infty} a_k e^{ikt}$.

Given $F(t)$, how do we find the c_k?

$$F(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt} \qquad \cdot e^{-iNt}$$

$$\rightsquigarrow \quad e^{-iNt} F(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt-iNt}$$

$$\rightsquigarrow \quad \int_{-\pi}^{\pi} e^{-iNt} F(t)\,dt = \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} c_k e^{ikt-iNt}\,dt$$

$$= \sum_{k=-\infty}^{\infty} c_k \int_{-\pi}^{\pi} e^{ikt-iNt}\,dt \qquad \left(\int_{-\pi}^{\pi} e^{ikt} e^{-iNt} \frac{dt}{2\pi} = \begin{cases} 1 & k=N \\ 0 & \text{otherwise} \end{cases} \right)$$

$$= 2\pi \cdot c_N$$

$$\therefore \quad \boxed{c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t)\, e^{-ikt}\,dt}$$

Defn These c_k are the Fourier coefficients of $F(t)$.
$\sum_{k=-\infty}^{\infty} c_k e^{-ikt}$ is the Fourier series of F.

Given any piecewise continuous $F(t)$ we define c_k and $\sum c_k e^{ikt}$ by the formulas above.

We will prove that the Fourier series converges (to $F(t)$) if $F(t)$ is piecewise continuous. This is a priori not obvious!

ex $F(t) = e^{2\cos t}$

Computation of the c_k:

$$F(t) = \underbrace{e^{z+\frac{1}{z}}}_{=:f(z)} \qquad z = e^{it}$$

$$f(z) = e^{z+\frac{1}{z}} = \left(\sum_{m=0}^{\infty} \frac{z^m}{m!}\right)\left(\sum_{l=0}^{\infty} \frac{z^{-l}}{l!}\right)$$

$$= \sum_{n=-\infty}^{\infty} z^n \underbrace{\left(\sum_{m=|n|} \frac{1}{m!} \cdot \frac{1}{(m-|n|)!}\right)}_{=:c_n}$$

$$= \sum_{n=-\infty}^{\infty} c_n z^n$$

$\therefore$ $F(t) = \sum_{n=-\infty}^{\infty} c_n e^{inz}$, where $c_n = \sum_{m=|n|} \frac{1}{m!} \frac{1}{(m-|n|)!}$

would have been hard to see this using integration..

ex $F(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$

$$c_0 = \int_{-\pi}^{\pi} F(t) \frac{dt}{2\pi} = 0.$$

$k \neq 0$: $$c_k = \frac{-1}{2\pi}\int_{-\pi}^{0} e^{-ikt}\,dt + \frac{1}{2\pi}\int_0^{\pi} e^{-ikt}\,dt$$

$$= \frac{e^{-ikt}}{2\pi i k}\Big|_{-\pi}^{0} - \frac{e^{-ikt}}{2\pi i k}\Big|_0^{\pi} = \frac{1}{2\pi i k}\left[1 - (-1)^k - (-1)^k + 1\right]$$

$$= \frac{1}{\pi i k}\left[1 - (-1)^k\right] = \begin{cases} 0 & k \text{ even} \\ \frac{2}{\pi k i} & k \text{ odd} \end{cases}$$

$\qquad\qquad\qquad$ will justify later

$$\therefore \; F(t) = \frac{2}{\pi i} \sum_{k \text{ odd}} \frac{1}{k} e^{ikt}$$

$$= \cdots + \left(\frac{-2}{3\pi i}\right) e^{-3it} + \left(\frac{-2}{\pi i}\right) e^{-it} + \left(\frac{2}{\pi i}\right) e^{it} + \frac{2}{3\pi i} e^{3it} + \cdots$$

$$= \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right)$$

Warning: This does not converge uniformly, since the sum is not continuous.

Thm. If $F(t) = f(e^{it})$ is piecewise continuous (or L^2) with Fourier coefficients c_k, then for $m, n \geq 0$

$$\sum_{k=-m}^{n} |c_k|^2 + \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\left| F(t) - \sum_{k=-m}^{n} c_k e^{ikt} dt \right|}_{\text{"error term"}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |F(t)|^2 dt$$

Proof.

$$\left| F(t) - \sum_{k=-m}^{m} c_k e^{ikt} dt \right|^2 = \left(F(t) - \sum_{j=-m}^{n} c_j e^{ijt} \right)\left(\overline{F(t)} - \sum_{k=-m}^{n} \overline{c_k} e^{-ikt} \right)$$

Integrate from $-\pi$ to π on both sides: Only the terms with $j=k$ survive.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(t)|^2 - \underbrace{\sum \overline{c_k} \int_{-\pi}^{\pi} F(t) e^{-ikt} \frac{dt}{2\pi}}_{\overline{c_k} c_k} - \underbrace{\sum c_j \int_{-\pi}^{\pi} \overline{F(t)} e^{ijt} \frac{dt}{2\pi}}_{c_j \overline{c_j}} + \sum |c_j|^2$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |F(t)|^2 dt - \sum |c_k|^2 - \cancel{\sum |c_j|^2} + \cancel{\sum |c_j|^2}$$

$\Rightarrow$ OK. $\square$

This shows that the partial sums $\sum_{k=-m}^{n} |c_k|^2$ are bounded.

$\Rightarrow$ the series $\sum_{k=-m}^{n} |c_k|^2$ converges

Theorem (Bessel's inequality) If $F(t) = f(e^{it})$ is piecewise continuous with Fourier series $\sum c_k e^{ikt}$, then

$$\sum_{k=-\infty}^{\infty} |c_k|^2 \leq \frac{1}{2\pi}\int_{\pi}^{\pi} |F(t)|^2 dt$$

Thm Suppose $F(t) = f(e^{it})$ is piecewise continuous with Fourier series $\sum c_k e^{ikt}$.

If $f(e^{it})$ is differentiable at $t = t_0$, then the Fourier series of $f(e^{it})$ converges to $f(e^{it_0})$ at $t = t_0$:

$$f(e^{it_0}) = \sum_{k=-\infty}^{\infty} c_k e^{ikt_0}.$$

Proof. ① Reduce to the case $t_0 = 0$ (and $e^{it_0} = 1$)

Let $h(t) = f(e^{i(t+t_0)})$ and let $\sum a_k e^{ikt}$ be the Fourier series.

piecewise continuous and differentiable at $t = 0$

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i(t+t_0)}) e^{-ikt} dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u) e^{-iku} e^{ikt_0} = c_k \cdot e^{ikt_0}$$

∴ The Fourier series of h (at $t=0$) is the same as that of $f(e^{it})$ (at $t = t_0$). ⟶ If we prove it for h we get it for f.

② Completion of the proof.

Let $g(e^{it}) = \dfrac{f(e^{it}) - f(1)}{e^{it} - 1}$

f differentiable at $t = 0$ ⟶ $\lim_{t \to 0} g(e^{it})$ exists and g also piecewise continuous.

(b_k) = Fourier coefficients of $g(e^{it})$.

Bessel's inequality ⟶ $b_k \xrightarrow[k \to \pm\infty]{} 0$

Note:
$$f(e^{it}) = g(e^{it})\,(e^{it}-1) + f(1)$$

$$\rightsquigarrow \quad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{it})\,(e^{it}-1)\,e^{-ikt}\,dt + \frac{f(1)}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum b_k e^{ikt}\right)(e^{it}-1)\,e^{-ikt}\,dt + \frac{f(1)}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\,dt$$

$$\Rightarrow c_k = b_{k-1} - b_k \qquad k \neq 0$$

$$c_0 = b_{-1} - b_0 + f(1)$$

$\Rightarrow$ $\sum c_k$ telescopes !

$$\Rightarrow \sum_{k=-m}^{n} c_k = f(1) + \sum_{k=-m}^{n} (b_{k-1} - b_k) = f(1) + \overbrace{b_{-m-1} - b_n}^{\text{small!}}$$

$$\xrightarrow[m,n\to\infty]{} f(1)$$

This proves the theorem!

Note: For $f(e^{it}) = \begin{cases} 1 & 0 < t < \pi \\ -1 & -\pi < t < 0 \end{cases}$ this does not give information about $t=0$. However, the formula still converges at $t=0$.

Thm $f(e^{it})$ a C^1 function in t, with Fourier series $\sum c_k e^{ikt}$

Then the Fourier series of $\frac{d}{dt} f(e^{it})$ is obtained as

$$\frac{d}{dt} f(e^{it}) \sim \sum ik c_k e^{ikt}$$

proof

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial}{\partial t} f(e^{it})\, e^{-ikt}\, dt = \frac{e^{-ikt}}{2\pi} f(e^{it})\Big|_{-\pi}^{\pi} + \frac{ik}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\, e^{-ikt}\, dt$$

integration by parts

$$= ik c_k \qquad \square$$

$\therefore$ the Fourier coefficients of $\frac{d}{dt} f(e^{it})$ decrease at least as rapidly as $\frac{1}{k}$ (By Bessel's inequality: $\sum |k c_k|^2$ must converge) $\Rightarrow k c_k \to 0$

induction $\rightsquigarrow$ The Fourier coefficients of an n-times differentiable function $f(e^{it})$ decrease at least as rapidly as $\frac{1}{k^n}$.

Cor If $f(e^{it})$ is C^n with Fourier coefficients c_k

then

$$\sum_{k=-\infty}^{\infty} k^{2n} |c_k|^2 < \infty, \quad \text{and} \quad k^n c_n \xrightarrow[k\to\infty]{} 0$$

($\frac{d^n}{dt^n} f(e^{it})$ has Fourier coefficients $(ik)^n c_k e^{ik\theta}$)

$\rightsquigarrow$ Apply Bessel's inequality

Thm Suppose $f(e^{it})$ is C^2. Then the Fourier series converges uniformly as a function of t.

Proof. We have that

$$k^2 c_k \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \pm\infty$$

$\Rightarrow$ $\sum |c_k|$ converges (by Comparison test with $\sum \frac{1}{k^2}$)

$\Rightarrow$ $\sum c_k e^{ikt}$ converges uniformly (to a function of t)

M-test

$\Rightarrow$ $\sum c_k e^{ikt}$ converges uniformly to $f(e^{it})$ $\square$

pointwise

Extended example

$f(e^{it}) = t^2 \quad -\pi \le t \le \pi$

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{1}{2\pi}\left[\frac{t^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 e^{-ikt}\,dt \qquad k \neq 0$$

$$= \frac{1}{2\pi}\left[e^{-ikt}\left(\frac{-2i + 2kt + ik^2t^2}{k^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left(e^{-ikt}\frac{2kt}{k^3}\right)\Big|_{-\pi}^{\pi} = \frac{1}{2\pi}\left((-1)^k\cdot\frac{2\pi}{k^2} - (-1)^k\cdot\frac{2\cancel{k}(-\pi)}{k^2}\right)$$

$e^{-ik\pi t} = (-1)^k$

(M-test)

$$= \frac{(-1)^k}{2\pi}\left(\frac{2\pi}{k^2} + \frac{2\pi}{k^2}\right) = \frac{2(-1)^k}{k^2} \qquad k \neq 0$$

t^2 is $C^2 \xrightarrow{1}$ the series converges uniformly on $(-\pi, \pi)$

$$t^2 = \frac{\pi^2}{3} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{2(-1)^k}{k^2} e^{-ikt} = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} e^{-ikt}$$

$t = 0$

$$\therefore \quad 0 = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} \qquad \therefore \quad \frac{\pi^2}{12} = -\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}$$

Let $\sum_{k=1}^{\infty}\frac{1}{k^2} \Rightarrow$

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

$$= \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right) + \underbrace{2\left(\frac{1}{2^2} + \frac{1}{4^2} + \ldots\right)}_{= \frac{1}{2}\zeta(2)}$$

$$= \frac{\pi^2}{12} + \frac{1}{2}\zeta(2)$$

$$\Rightarrow \quad \zeta(2) = 2\cdot\frac{\pi^2}{12} = \frac{\pi^2}{6}.$$

The Dirichlet problem and harmonic extensions

Q: Given a domain $U \subseteq \mathbb{C}$, and a continuous function h on ∂U, can one construct an extension $\tilde{h} : U \to \mathbb{C}$ s.t $\tilde{h}$ is harmonic and $\tilde{h}|_{\partial U} = h$?

We first consider the case where $U = \{|z| \leq 1\}$, so $h : S^1 \to \mathbb{C}$ is defined on the unit circle.

Consider the case where $h(e^{i\theta}) = \sum_{k=-N}^{N} a_k e^{ik\theta}$ is a trigonometric polynomial.

In this case
$$\tilde{h}(re^{i\theta}) = \sum_{k=-N}^{N} a_k r^{|k|} e^{ik\theta}$$
works:

- $\tilde{h}|_{r=1} = h$
- $\tilde{h}$ is a sum of terms of the form z^k, $\bar{z}^{-k}$ $\quad (z = re^{i\theta})$
  $\to \tilde{h}$ harmonic

$\tilde{h}$ is also unique, by the maximum principle (two extensions are zero on ∂U $\Rightarrow$ zero on all of U)

We can solve for the a_m:
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\theta} h(e^{i\theta}) = \dots = a_m$$

$$\Rightarrow \quad \tilde{h}(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(e^{i\varphi}) \underbrace{\left[\sum_{k=-\infty}^{\infty} r^{|k|} e^{-ik\varphi} e^{ik\theta}\right]}_{=: P_r(\theta)} d\varphi$$

Defn $P_r(\theta)$ is the Poisson kernel.

This converges uniformly (by the M-test) on $r \le \rho < 1$, $-\pi \le \theta \le \pi$.

$$\therefore \quad \tilde{h}(re^{i\theta}) = \int_{-\pi}^{\pi} h(e^{i\varphi})\, P_r(\theta-\varphi)\, \frac{d\varphi}{2\pi} = \int_{-\pi}^{\pi} h(e^{i(\theta-\varphi)})\, P_r(\varphi)\, \frac{d\varphi}{2\pi}$$

Simplified expression for $P_r(\theta)$: $\quad z = re^{i\theta}$

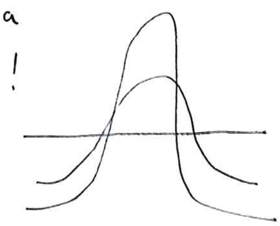
$$\Rightarrow \quad P_r(\theta) = 1 + \sum_{k\ge 0} z^k + \sum_{k \ge 1} \bar{z}^k = 1 + \frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}}$$

$$= \frac{1-|z|^2}{|1-z|^2} = \frac{1-r^2}{1+r^2-2r\cos\theta} \qquad (*)$$

Some properties:

- $P_r(\theta) > 0$
- $P_r(-\theta) = P_r(\theta)$
- $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1$

For r fixed, $P_r(\theta)$ is a probability measure!



- $P_r(\theta)$ increasing $-\pi \le \theta \le 0$, decreasing $0 \le \theta \le \pi$
- For $\delta > 0$, $\max\{ P_r(\theta) : \delta \le |\theta| \le \pi \} \xrightarrow[r \to 0]{} 0$
- $P_r(\theta)$ is harmonic $\left(= \operatorname{Re}\left(\frac{1+z}{1-z}\right)\right)$

analytic in $\{|z|<1\}$

Now, given any h continuous on S^1, define $P_r(\theta)$ and $\tilde{h}$ via the formulas above.

Thm $h(e^{i\theta})$ continuous function on S^1. Then

$\tilde{h}$ is harmonic on $\{|z|<1\}$ and $\tilde{h}(z) \xrightarrow[z \to \partial U]{} h(z)$.

proof
- harmonicity $\rightsquigarrow$ just differentiate under $\int$.
- $\tilde{h} \to h$ $\rightsquigarrow$ uniformly approximate h by $\sum a_m e^{im\theta} =: g(\theta)$

$|h(e^{i\theta}) - g(e^{i\theta})| \le \varepsilon \Rightarrow |\tilde{h}(z) - \tilde{g}(z)| < \varepsilon \quad \forall z$ (maximum principle)

$\Rightarrow$ $\tilde{h}(z)$ tends to $\tilde{g}(z)$ which tends to $g(z)$ on ∂U which tends to $h(e^{i\theta})$

$\longrightarrow$ OK.

Finally, we can prove our fundamental theorem for harmonic functions:

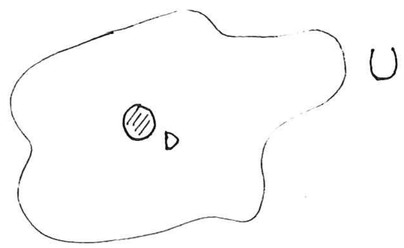
Thm (Characterisation of harmonic functions)

$h: U \longrightarrow \mathbb{C}$ continuous function on a domain U

Then $h(z)$ is harmonic if and only if $h(z)$ has the mean value property.

(Recall: $\forall z_0 \in U$
$h(z_0) =$ average of $h(z)$ over a small disk around z_0)

Proof



Given h $\rightsquigarrow$ $\exists$ harmonic function $g: D \to \mathbb{C}$ s.t. $g|_{\partial D} = h(z)|_{\partial D}$.

$\Rightarrow$ $h(z) - g(z)$ continuous on D which satisfies the mean value property

$\Rightarrow$ $h(z) - g(z) = 0 \quad \forall z \in D$ (since $g(z) \to h(z)$ on ∂D)

$\Rightarrow$ $h(z) = g(z)$ is harmonic in D

$\Rightarrow$ $h(z)$ is harmonic in all of U.

This is surprising: A continuous function with the mean value property automatically has partial derivatives of all orders!

Cor The uniform limit of harmonic functions is again harmonic

proof The limit is continuous, and has the mean value property. □