The Argument Principle

Set-up: $f(z)$: some meromorphic function

γ: some closed curve in $\mathbb{C}$

Q: How many zeroes / poles are there inside γ?

The Argument principle gives a method to count this.

Defn Suppose f is analytic and $f(z) \neq 0$ on γ. We call

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i} \int_\gamma d\left(\log f(z)\right)$$

the logarithmic integral of $f(z)$ along γ.

something you can compute! (e.g numerically)

Ex1 $f(z) = (z-a)^n$ $\quad a \in \mathbb{C}$, $n \in \mathbb{Z}$

$\gamma(\theta) = a + e^{i\theta}$ $\quad \theta \in [0, 2\pi k]$

Note that γ winds around a k times..

$$\frac{f'(z)}{f(z)} = \frac{n(z-a)^{n-1}}{(z-a)^n} = \frac{n}{z-a}$$

$\therefore$ non-zero + analytic on γ

$$\therefore \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i} \int_\gamma \frac{n\, dz}{z-a} \underset{\substack{z = e^{i\theta} + a \\ dz = ie^{i\theta} d\theta}}{=} \frac{n}{2\pi i} \int_0^{2\pi k} \frac{i e^{i\theta}\, d\theta}{e^{i\theta}} = \frac{n}{2\pi} \int_0^{2\pi k} d\theta = \underline{n \cdot k}$$

Note: f has a zero of order n inside γ.

Essentially the same computation shows

<u>Thm</u> (Argument principle I) $D \subseteq \mathbb{C}$ bounded domain with smooth ∂D

$f(z)$ meromorphic on D + analytic on ∂D, + $f(z) \neq 0$ on ∂D.

$$\Rightarrow \quad \frac{1}{2\pi i} \int \frac{f'(z)}{f(z)}\, dz = N_0 - N_\infty$$

$N_0 = \#$ zeros

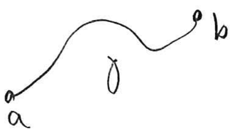
$N_\infty = \#$ poles

counted with multiplicities.

Note: $\frac{f'(z)}{f(z)} dz = d(\log f(z))$. $\quad \log f(z) = \log|f(z)| + i \arg f(z)$

$= d \log|f(z)| + i\, d \arg f(z)$

$\int_\gamma d \log|f(z)| = \log|f(b)| - \log|f(b)| = 0$ when γ is closed



$\therefore$ It is only the change in arg that contributes to $N_0 - N_\infty$.

<u>Thm</u> (Argument principle II) Same hypotheses as above

$$\int_{\partial D} d \arg f(z) = 2\pi (N_0 - N_\infty)$$

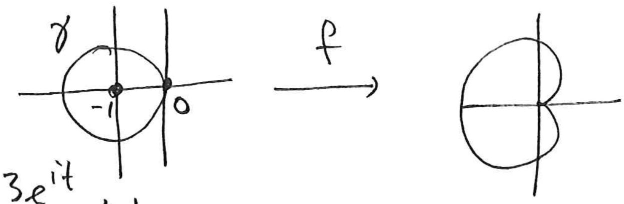
ex) $f(z) = z^3 + 1$

Ⓐ $\gamma(t) = 2e^{it}$ $t \in [0, 2\pi]$

$f(\gamma(t)) = 8e^{3it} + 1$ $\therefore$ f loops around the origin <u>3 times</u>.

$\therefore$ $N_0 - \underset{0}{\underset{\shortparallel}{N_\infty}} = 3$ $\therefore$ $N_0 = 3$

Ⓑ $\gamma(t) = e^{it} - 1$



$f(\gamma(t)) = e^{3it} - 3e^{2it} + 3e^{it} + 1$

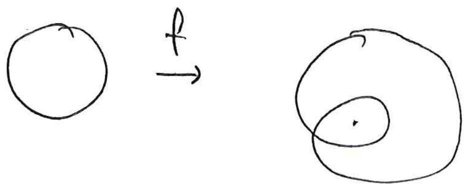
$\therefore$ $f(\gamma(t))$ increases its argument by 2π

$\rightarrow$ only one zero falls inside γ.

ex) $f(z) = e^z - 2z$

$\gamma(t) = \cancel{R}\ 3e^{it}$ $t \in [0, 2\pi]$

$f(\gamma(t)) = e^{3e^{it}} - 3e^{it}$

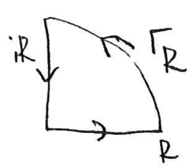


winds around twice $\Rightarrow$ two roots

ex)

$f(z) = z^4 + 2z^2 - z + 1$

γ:



$\int_0^R \arg f(x)\, dx$

For real x: $2x^2 - x + 1 = (x - \frac{1}{2})^2 + x^2 + \frac{3}{4} > 0$

$x^4 + 2x^2 - x + 1 = x^4 + (x - \frac{1}{2})^2 + x^2 + \frac{3}{4} > 0$

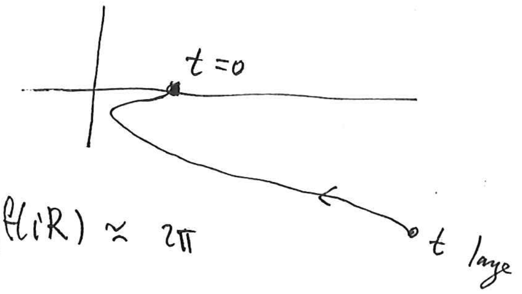
(check using calculus!)

$\therefore$ $f(x)$ is always a positive real value $\therefore$ $\int_0^R d\arg f(x) = 0$

$$\int_{\Gamma_R} d\arg f(x) \underset{R\to\infty}{\approx} \int_R d\arg z^4 = 4\cdot\frac{\pi}{2} = \underline{2\pi}.$$

$$\int_{iR}^{0} d\arg f(x) = -\int_0^R d\arg f(it)$$

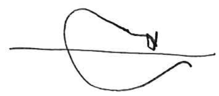
$$f(it) = (it)^4 + 2(it)^2 - (it) + 1 = t^4 - 2t^2 + 1 + -it$$



initially $\arg f(iR) \approx 2\pi$

Need to check whether it changes quadrant:

$\Leftrightarrow \nexists\ t\in(0,1)$ s.t. $\operatorname{Im} f(it) = 0 \Leftarrow$ OK.



$$\therefore -\int_0^R d\arg f(it) = 0$$

$$\therefore \int_\gamma d\arg f(z) = 2\pi = 2\pi(N_0 - N_\infty)$$

$\Rightarrow$ exactly one zero in first quadrant.

Can use this to integrate:

$$\int_{|z-1|=\varepsilon} \frac{z^2\,dz}{z^3+1} = \frac{1}{3}\int_{|z-1|=\varepsilon} \frac{3z^2}{z^3+1} = \frac{1}{3}\left[2\pi i\,(N_0 - N_\infty)\right]$$

$\varepsilon > 0$ small
$\Rightarrow z^3+1$ has no zeroes/poles in $|z| < \varepsilon$.

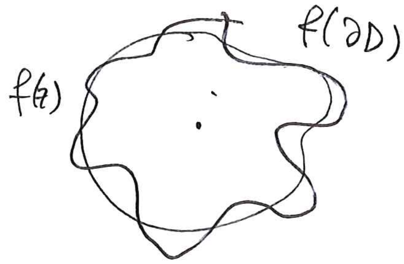
$$= \frac{1}{3}\cdot 2\pi i \cdot 1$$

$$= \underline{\frac{2\pi}{3} i}$$

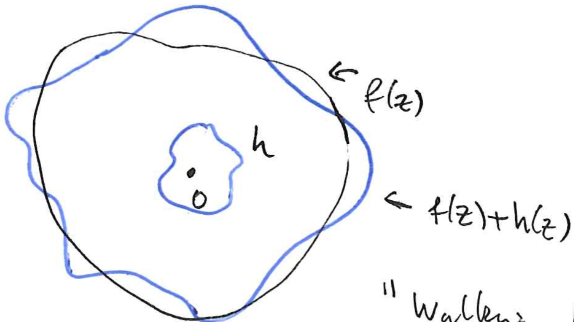
Rouché's theorem $D \subseteq \mathbb{C}$ bounded domain

∂D smooth f, h analytic on $\overline{D} = \partial D \cup D$

If $|h(z)| < |f(z)|$ for $z \in \partial D$, then $f(z)$ and $f(z) + h(z)$ have the same number of zeroes (counting multiplicities).



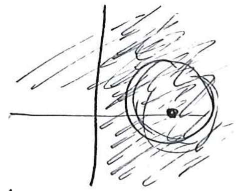
$f(z) + h(z)$ is a perturbation of $f(z)$:



$\therefore$ $f(z) + h(z) = 0$ then $f(z)$ has to be "cancelled" by $h(z)$ $\Rightarrow$ not possible.

"Walking the dog around the pole theorem"

$$|h(z)| < |f(z)| \Rightarrow \left|\frac{h(z)}{f(z)}\right| < 1 \Rightarrow \mathrm{Re}\left(1 + \frac{h(z)}{f(z)}\right) > 0 \Rightarrow$$



values in half space $\to$ single valued arg.

$$\arg(f(z) + h(z)) = \arg f(z) + \arg\left(1 + \frac{h(z)}{f(z)}\right)$$

$\therefore$ change in is matched by change in

$\xrightarrow[\text{principle}]{\text{argument}}$ # zeroes of $f+h$ = # zeros of f.

This is good for computations: Write $P(z) = \underset{}{\text{BIG}}^{\,= f(z)} + \text{little}^{\,= h(z)}$

If $f(z)$ is simpler and one can show $|h(z)| < |f(z)|$

$\rightarrow$ get information about $P(z)$ from $f(z)$.

ex1 $p(z) = z^6 - 5z^5 + z^3 - 2 = 0$ in $|z| \le 1$

$f(z) = -5z^5$ ← has a zero of multiplicity 5

$h(z) = z^6 + z^3 - 2$

$|f(z)| = 5$

$|h(z)| \le 1 + 1 + 2 = 4$

$\therefore |h(z)| < |f(z)|$ $\underset{\text{Rouche}}{\Rightarrow}$ $P(z) = f(z) + h(z)$ has 5 roots in $|z| = 1$

finding the right $h(z)$ requires a little bit of practice..

ex1 $p(z) = a_n z^n + a_{n-1} z^{n-1} + .. + a_0$

$f(z) = a_n z^n$ ← has a zero of order n

on $|z| = R$ $h(z) = a_{n-1} z^{n-1} + .. + a_0$

$|h(z)| \le |a_{n-1}| R^{n-1} + .. + |a_0| \le R^n |a_n|$ since $n \ge 1$.

$R >> 0$

$\therefore p(z) = f(z) + h(z)$ has n zeroes (multiplicity).

ex1 $3e^z - z = 0$ in $|z| \le 1$

$f(z) = 3e^z$ no zeroes

$h(z) = -z$

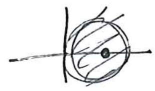
$z = x + iy$ $x^2 + y^2 \le 1$

$|h(z)| \le 1 \le |3e^z| = |3e^x e^{iy}| = |3e^x| = 3e^x \ge 3e^{-1} \ge 1.$

$\Rightarrow$ no solutions.

ex1 $\quad |\lambda| < 1$

$(z-1)^n e^z = \lambda$ has n solutions in $|z-1| < 1$

and no other solutions in  $\operatorname{Re} z > 0$

$f(z) = (z-1)^n e^z$ has n solutions in $|z-1|<1$

$h(z) = \lambda$

$z = x+iy$

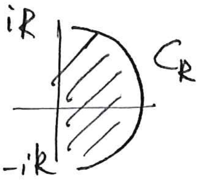
$|h(z)| = |\lambda| < 1$ $\qquad |z-1|=1$ $\qquad |e^x \cdot e^{iy}| = \underline{e^x > e^0} = 1$

$|e^z| > 1 > |h(z)|$

$|f(z)| = |z-1|^n |e^z| \geq \cancel{|z-1|^n} \geq |h(z)|$

$\Rightarrow$ f has n solutions in $|z-1|<1$

No others: $\qquad x > 0$

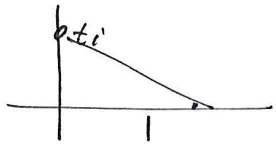


$|f(z)| \geq \cancel{1} > |\lambda|$ for $z \in \partial C_R$

$|z| = R \qquad r = \sqrt{R} > 0$

$(x^2 - 2x + 1 + y^2)^{n/2} \qquad |(x^2 - 2x + 1 + y^2)^{n/2}| = (R - 2x + 1)^{n/2} > 1$ for $R \gg 0$.

$z = it \qquad t \in [-R, R]$.



$|f(z)| = |(ti-1)^n e^{iz}| = |(ti-1)^n| = |ti-1|^n$

$= (t^2+1)^{n/2} > 1$ ok.

$|z| = r^2 = R$

$|f(z)| = |z-1|^{n/2} |e^z| > (|z|-1)^{n/2} |e^{\cancel{z}}| > 1$ for $R \gg 1$

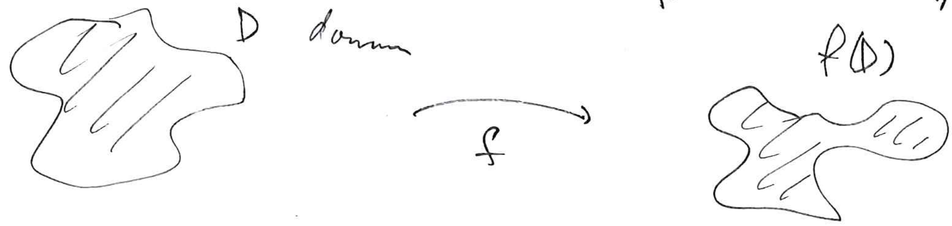
$\quad\quad (R-1)^{n/2} e^R >$

Open mapping Theorem

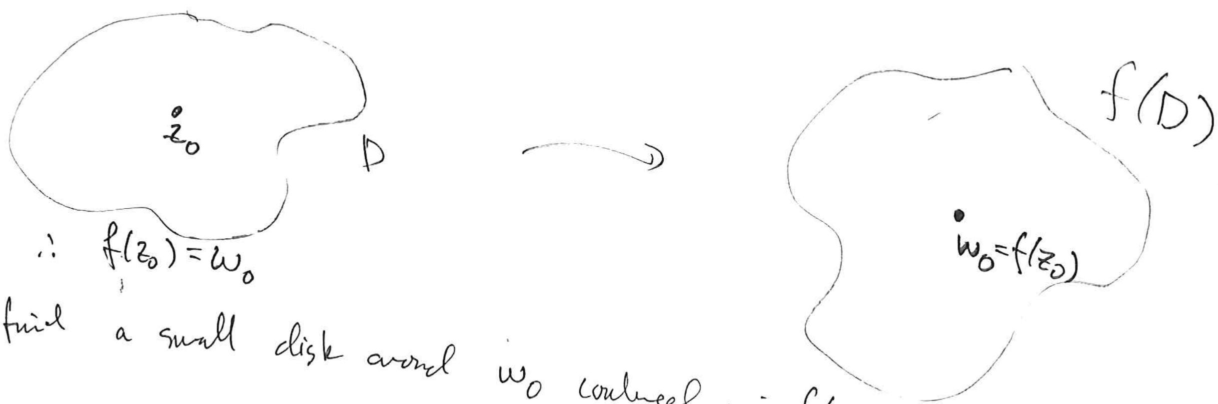
This fails for simple real functions:

$f(x)=x^2 \quad f(-1,1) = [0,1) \leftarrow$ not open

$f(x)=\sin x \quad f(-3\pi, 3\pi) = [-1,1] \leftarrow$ not open.



If f is analytic + non-constant, then $f(z)$ maps open sets to open sets : $f(U)$ open for U open.



$w_0 \in f(D) \quad \therefore \quad f(z_0)=w_0$

Want to find a small disk around w_0 contained in $f(D)$.

$f(z)-w_0$ isolated zeroes $\Rightarrow$ can pick $\rho>0$ s.t $f(z)-w_0 \neq 0$ for $0<|z-z_0|\leq\rho$

Let $|f(z)-w_0| \geq \delta$ for $|z-z_0|=\rho$.

Consider

$$N(w) = \frac{1}{2\pi i}\int_{|z-z_0|=\rho} \frac{f'(z)}{f(z)-w}\,dz \quad \text{on} \quad |w-w_0|<\delta$$

well defined + analytic in w

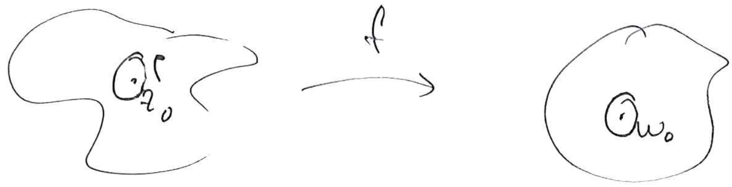
$= \#$ zeroes of $f(z)-w$ in $|z-z_0|<\rho$

$N(w)$ analytic + integer-valued $\Rightarrow$ constant

$\therefore$ If $f(z_0)=w_0$ has m solutions $\Rightarrow$ $f(z)=w$ has m solutions for w in a small nbh of w

$\therefore f(D)$ is open.

Suppose $f(z)-w_0$ has a simple zero at $z=z_0$.



$\therefore$ f is 1-1 in a nbh of z_0 (since $N(w)=1$ there)

$\rightarrow$ there should be an inverse function of f here.

Thm (Inverse function theorem)

Define

$$g(w) = \frac{1}{2\pi i}\int_{|z-z_0|=\rho} \frac{z f'(z)}{f(z)-w}\,dz$$

analytic function of z for $|z-z_0|\le\rho$ except for a simple pole at $z=z_0$

Then $g(w)=f^{-1}(w)$ is the inverse function of f; it is analytic in w.

$$\mathrm{Res}\left(\frac{z f'(z)}{f(z)-w}, z_0\right) = \lim_{z\to z_0}\left((z-z_0)\frac{z f'(z)}{f(z)-w}\right) = \frac{f'(z)z}{f'(z)}\Big|_{z=z_0} = z_0$$

$\therefore$ The residue theorem does this part.

This formula is not so important $\rightarrow$ important point: f^{-1} is analytic and $\exists$ formula for the Taylor series.

Maximum modulus principle

"f attains its maximum on ∂D" ✗

$D \subseteq \mathbb{C}$ ~~connected~~ domain

f analytic on D

If $\exists z_0 \in D$ s.t. $|f(z_0)| \geq |f(z)|$ $\forall z \in D$ $\rightarrow$ f constant.

proof Suppose such z_0 exist, and f is non constant.

$\Rightarrow$ $f(D)$ open set

$\Rightarrow$ $\exists w \in f(D)$ with $|w| > |f(z_0)|$ $w = f(z_1)$!

contradicting maximality of f

$\Rightarrow$ proof complete.

Max modulus theorem

$$\sup_{\overline{D}} |f| = \sup_{\partial D} |f| \leftarrow (*)$$

$f : D \to \mathbb{C}$ open map. If $(*)$ fails then $\exists z_0 \in D$ s.t.

$|f(z_0)| \geq |f(z)|$ for all $z \in \overline{D}$

$\exists \delta$ s.t. $D(z,\delta) \subset \overline{D(z,\delta)} \subset D$.

$f : \partial D(z_0,\delta) \rightarrow \partial f(D(z_0,\delta))$

$\max \{ |w| : w \in \partial f(D(z_0,\delta)) \} >$

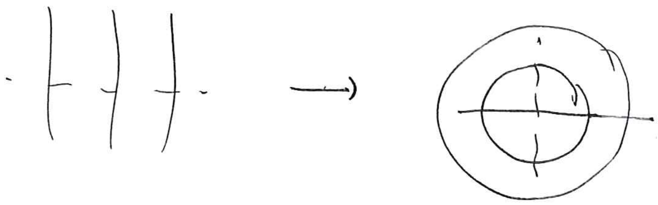
Exp

$$z = x+iy$$

$$e^z = e^x(\cos y + i\sin y)$$

$$= e^x e^{iy}$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!}$$



Log

$$\mathrm{Log}\, z = \log|z| + i\,\mathrm{Arg}\, z \qquad (\text{Recall: } \mathrm{Arg}\, z \in (-\pi, \pi])$$

$$\log z = \log|z| + i\arg z$$

$$= \log|z| + i\,\mathrm{Arg}\, z + 2\pi i k \quad \leftarrow \text{multivalued}$$

$$\mathrm{Log}(1+i) = \log\sqrt{2} + i\frac{\pi}{4}$$

$|1+i| = \sqrt{2}$

$\mathrm{Arg}\,|1+i| = \pi/4$

$$\mathrm{Log}\, i = i\frac{\pi}{2}$$

Power $\alpha \in \mathbb{C}$

$$z^\alpha = e^{\alpha \log z} \quad \leftarrow \text{multivalued}$$

$$= e^{\alpha\log|z| + \alpha i\,\mathrm{Arg}\, z + 2\pi i k\alpha}$$

$$i^{-i} = e^{-i\log i} = e^{-i\cdot i\cdot\pi/2 + 2\pi i k\cdot i} = e^{\pi/2 + 2\pi k}.$$

$$\mathrm{Log}\, i = i\frac{\pi}{2}$$

ML estimate

$$\left| \int_\gamma f(z)\,dz \right| \le ML \qquad M = \sup_\gamma |f(z)|$$
$$L = \text{arc length of } \gamma$$

Cauchy's theorem

D bounded domain with smooth ∂D + smooth extension to ∂D

$$\int_{\partial D} f(z)\,dz = 0 \quad \forall \text{ analytic } f.$$

(follows from Green's theorem)

(and f analytic $\Leftrightarrow$ $\int_{\square} f(z) = 0$)

Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\,dw \qquad z \in D$$

$$f^{(m)}(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}\,dw$$

Cauchy estimates

If $|f(z)| \le M \;\; \forall\, \cancel{z\in D}\; |z-a| = \rho$

then $\left| f^{(m)}(z) \right| \le \dfrac{m!}{\rho^m} M$ for any $m > 0$

In particular $|f'(z)| \le \dfrac{M}{\rho}$.

Liouville's theorem

$f: \mathbb{C} \to \mathbb{C}$ analytic on all of $\mathbb{C}$ (f entire)

If f is bounded, then f is constant.

proof $\exists M > 0$ s.t $|f(z)| \leq M \quad \forall z \in \mathbb{C}$

On $|z-a| < \rho \Rightarrow |f'(a)| \leq \frac{M}{\rho}$ ← LHS indep of ρ

Now let $\rho \to \infty \Rightarrow |f'(a)| = 0 \Rightarrow f'(a) = 0 \Rightarrow f$ constant ~~[illegible]~~
(Since this holds for every a).

Note: $\frac{1}{1+x^2}$ is bounded on $\mathbb{R}$, but not constant.

Cor FTA

Cor If f, g are entire and $|f| \leq |g|$ then $f = \alpha \cdot g$ for some $\alpha \in \mathbb{C}$.

$g \equiv 0$ or
$g \not\equiv 0$: Let $h(z) = \frac{f(z)}{g(z)}$ ← $|h|$ is bounded by 1!
then if h can be extended to an entire function it has to be constant.

h analytic everywhere when $g(z) \neq 0$

z a zero of $g \Rightarrow z$ is isolated.

h bounded $\Rightarrow$ every singularity is removable $\Rightarrow$ OK.

f entire, then $f(\mathbb{C})$ is dense in $\mathbb{C}$. ($f(\mathbb{C})$ intersects any ~~ball~~ disk)

If not $\exists\, w \in \mathbb{C},\ r>0$ s.t $D(w,r) \cap f(\mathbb{C}) = \emptyset$

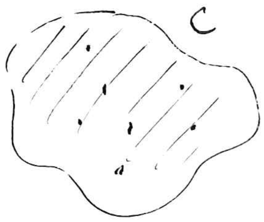
Define
$$g(w) = \frac{1}{f(z)-w}$$

$\Rightarrow$ g is entire, bounded: $|g(z)| = \dfrac{1}{|f(z)-w|} < \dfrac{1}{r}$

$\Rightarrow$ g constant

$\Rightarrow$ f constant.

The Residue Theorem and evaluation of real integrals.



C simple closed curve

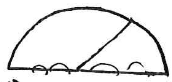
f analytic inside C and on C with a finite number of singularities $a_1 \dots a_k$

$$\therefore \int_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f, a_j)$$

Generalization of the previous formula $\frac{1}{2\pi i}\oint_C f(z)\,dz = a_{-1}$

Evaluation of definite integrals

① $\int_{-\infty}^{\infty} F(x)\,dx$ $F(x)$ a rational function.

Use a contour of the form 

If $F(x)$ is even $(F(-x) = F(x))$ then this can be used to evaluate $\int_0^{\infty} F(x)\,dx$

② $I = \int_0^{2\pi} G(\sin\theta, \cos\theta)\,d\theta$ G a rational function

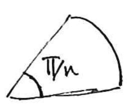
substitute $\sin\theta = \frac{z - z^{-1}}{2i}$ $\cos\theta = \frac{z + z^{-1}}{2}$ $dz = ie^{i\theta}\,d\theta$

$\rightarrow I = \int_C F(z)\,dz$ F rational function

③ $\int_{-\infty}^{\infty} F(x)\cos mx\,dx$ or $\int_{-\infty}^{\infty} F(x)\sin mx\,dx$ F rational function.

Again, use . Re, Im $\int_{-\infty}^{\infty} e^{imx} F(x)\,dx$.

④ Special contours:



sector

dogbone

Useful theorem: if $|F(z)| \le \frac{M}{R^k}$ $z = Re^{i\theta}$ $k>1$, then

$$\lim_{R\to\infty} \int_{\frown} F(z)\,dz = 0 \qquad \therefore \int_{\frown} F\,dz = \int_{-R}^{R} F\,dz.$$

Also,

$$\lim_{R\to\infty} \int_{\frown} e^{imz} F(z)\,dz = 0$$

ex1 $f(z) = e^z \csc^2 z$ double poles at $z = 0, \pm\pi, \pm 2\pi, \dots$

$$\operatorname{Res}(f, m\pi) = \lim_{z\to m\pi} \left((z-m\pi)^2 \frac{e^z}{\sin^2(z)} \right)' = \lim_{z\to m\pi} \left(e^z \frac{[(z-\pi m)^2 \sin^2 z + 2(z-m\pi)\sin z - 2\cos z\,(z-m\pi)^2]}{\sin^3 z} \right)$$

$$= e^{m\pi} \lim_{\substack{u\to 0\\ u = z-m\pi}} \frac{u^2 \sin u + 2u \sin u - 2u^2 \cos u}{\sin^3 u}$$

$$= e^{m\pi} \lim \left(\frac{u^2 \sin u + 2u\sin u - 2u^2 \cos u}{\sin^3 u} \right)$$

$$= \underline{e^{\pi m}}.$$

Much easier using Laurent series:

$$\underset{z = u+m\pi}{e^z \csc^2 z} = e^{m\pi + u}\csc^2 u = \frac{e^{m\pi}\left(1 + u + \frac{u^2}{2!} + \dots\right)}{\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \dots\right)^2}$$

$$= \frac{e^{m\pi}}{u^2}\left(1 + u + \frac{u^2}{2!} + \dots\right)\left(1 - \frac{u^2}{6} + \dots\right)^{-2}$$

$$= \frac{e^{m\pi}}{u^2}\left(1 + u + \frac{u^2}{2!} + \dots\right)\left(1 + \frac{u^2}{3!} + \dots\right)$$

$$= \frac{e^{m\pi}}{u^2}\left(1 + u + \frac{5}{6}u^2 + \dots\right) \qquad \therefore \operatorname{Res} = e^{m\pi}.$$

ex1

$$\int_C \underbrace{\frac{e^{zt}}{z^2(z^2+2z+2)}}_{=:f(z)} dz \qquad C : |z| = 3$$

f has a double pole at $z=0$
simple poles at $z=-1\pm i$ } these are inside C

$$\mathrm{Res}(f,0) = \lim_{z\to 0} \frac{1}{1!}\frac{d}{dz}\left(\cancel{z^2}\cdot\frac{e^{zt}}{\cancel{z^2}(z^2+2z+2)}\right) = \lim_{z\to 0}\left(\frac{(z^2+2z+2)(te^{zt}) - e^{zt}(2z+2)}{(z^2+2z+2)^2}\right)$$

$$= \frac{t-1}{2}$$

$$\mathrm{Res}(f,-1+i) = \lim_{z\to -1+i} (z-(-1+i))\frac{e^{zt}}{z^2(z^2+2z+2)}$$

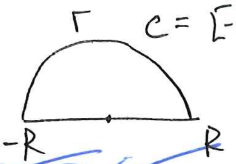
$$= \lim_{z\to -1+i}\frac{e^{zt}}{z^2}\cdot\lim_{z\to -1+i}\frac{z+1-i}{z^2+2z+2} = \frac{e^{(-1+i)t}}{(-1+i)^2}\cdot\frac{1}{2i} = \frac{e^{(-1+i)t}}{4}$$

$\mathrm{Res}(f,-1-i)$ = conjugate of the above $= \frac{e^{(-1-i)t}}{4}$

$$\therefore \int_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz = 2\pi i\left(\frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4}\right)$$

$$= 2\pi i\left(\frac{t-1}{2} + \frac{1}{2}e^{-t}\cos t\right)$$

ex) $I = \int_0^\infty \underbrace{\frac{dx}{x^6+1}}_{f(z)}$



$C = [-R,R] + \Gamma$

these lie in C

Simple poles at $z = e^{\pi i/6},\ e^{3\pi i/6},\ e^{5\pi i/6},\ e^{7\pi i/6},\ e^{9\pi i/6},\ e^{11\pi i/6}$

$$\mathrm{Res}(f, e^{\pi i/6}) = \lim_{z\to e^{\pi i/6}} \left((z - e^{\pi i/6}) \frac{1}{z^6+1} \right) \overset{L'H}{=} \lim_{z\to e^{\pi i/6}} \frac{1}{6z^5} = \frac{1}{6} e^{-5\pi i/6}$$

$$\mathrm{Res}(f, e^{3\pi i/6}) = \lim \left((z - e^{3\pi i/6}) \frac{1}{z^6+1} \right) = \lim_{z\to e^{3\pi i/6}} \frac{1}{6z^5} = \frac{1}{6} e^{-5\pi i/2}$$

$$\mathrm{Res}(f, e^{5\pi i/6}) = \lim_{z\to e^{5\pi i/6}} \frac{1}{6z^5} = \frac{1}{6} e^{-25\pi i/6}.$$

$$\therefore \int_C \frac{dz}{z^6+1} = 2\pi i \left(\frac{1}{6}\cdot e^{-5\pi i/6} + \frac{1}{6} e^{-5\pi i/2} + \frac{1}{6} e^{-25\pi i/6} \right)$$

$$= \underline{\frac{2\pi}{3}}$$

$$\therefore \int_{-R}^{R} \frac{dx}{x^6+1} + \underbrace{\int_\Gamma \frac{dz}{z^6+1}}_{\to 0 \text{ as } R\to\infty} = \frac{2\pi}{3}$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = 2\pi/3$$

$$\therefore \int_0^\infty \frac{dx}{x^6+1} = \underline{\frac{\pi}{3}}.$$

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_{|z|=1} \frac{dz/iz}{2+\frac{1}{2}(z+z^{-1})} = \frac{2}{i}\int_{|z|=1} \frac{dz}{z^2+4z+1}$$

$$\cos\theta = \frac{1}{2}(z+z^{-1})$$

$$d\theta = \frac{dz}{iz}$$

$$= 4\pi \operatorname{Res}\left(\frac{1}{z^2+4z+1}, \sqrt{3}-2\right)$$

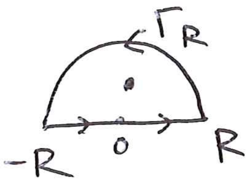
$$= \underline{\frac{2}{3}\pi\sqrt{3}}$$

ex1 $\quad I = \int_0^\infty \frac{\cos mx}{x^2+1}\,dx$

Note: $\quad I =$ Real part of $\int_0^\infty \frac{e^{imz}}{z^2+1}\,dz$

Let $f(z) = \frac{e^{imz}}{z^2+1}$; f has simple poles at $z = \pm i$.

$C_R = [-R, R] \cup \Gamma_R$ where:



$$\Rightarrow \int_{C_R} f\,dz = \int_{\Gamma_R} f\,dz + \int_{-R}^{R} f\,dz$$

← ~~this~~ this is what we are after

Jordan's lemma $\Rightarrow$ or ML estimate $\quad \lim_{R\to\infty} \int_{\Gamma_R} f\,dz = 0$

Also,
$$\int_{C_R} f(z)\,dz = 2\pi i \,\mathrm{Res}(f(z), i)$$
$$= 2\pi i \left(\lim_{z\to i} \frac{\cancel{\cos}\, e^{imz}}{(z+i)\cancel{(z-i)}} \cancel{(z-i)} \right)$$
$$= 2\pi i \left(\frac{e^{-M}}{2i} \right) = \pi e^{-M}$$

$$\Rightarrow \lim_{R\to\infty} \int f\,dz = \pi e^{-M}.$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos mx}{1+x^2}\,dx = \pi e^{-m} \qquad \therefore \int_0^\infty \frac{\cos mx}{1+x^2}\,dx = \underline{\underline{\frac{\pi}{2e^m}}}$$

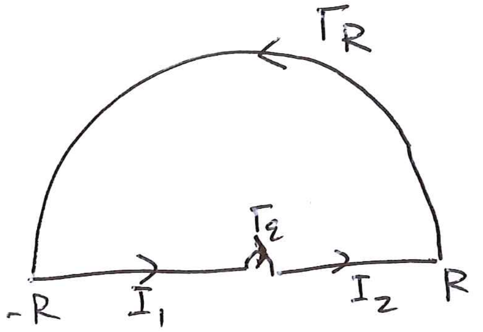
$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$$

We will present two proofs of this one.

① <u>Proof 1</u>

I = Imaginary part of $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = \int_{-\infty}^{\infty} \frac{\cos x}{x}\,dx + i\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$

Let $f(z) = \dfrac{e^{iz}}{z}$ and let C_R be the following contour:



$$\int_{C_R} f = \underbrace{\int_{-R}^{-\varepsilon} f}_{A} + \underbrace{\int_{\Gamma_R} f}_{B} + \underbrace{\int_{\varepsilon}^{R} f}_{C} + \underbrace{\int_{\Gamma_R} f}_{D}$$

when $\varepsilon \to 0$, both A and C give us $\displaystyle\int_0^R \frac{e^{ix}}{x}\,dx$.

D: This tends to zero as $R \to \infty$ by Jordan's lemma.

B: $$\int_{\Gamma_\varepsilon} \frac{e^{iz}}{z}\,dz = i(-\pi)\,\mathrm{Res}\left(\frac{e^{iz}}{z},\ z=0\right)$$ (Fractional residue theorem)

$$= -i\pi \cdot 1 = \underline{-i\pi}$$

Here is a direct proof (without the fractional residue theorem) of this fact:

negative direction!

$$z = \varepsilon \cdot e^{i\theta} \qquad dz = i\varepsilon e^{i\theta} d\theta \longrightarrow \int_{\Gamma_\varepsilon} f = -\int_0^{\pi} \frac{1 + (\varepsilon e^{i\theta}) + \frac{(\varepsilon e^{i\theta})^2}{2!} + \dots}{\cancel{\varepsilon} e^{i\theta}} i \cancel{\varepsilon} d\theta$$

$$= -\int_0^{\pi} i \left(1 + \varepsilon e^{i\theta} + \frac{(\varepsilon e^{i\theta})}{2!} + \dots\right)$$

uniform convergence

$$\xrightarrow[\varepsilon \to 0]{} -\int_0^{\pi} i \, d\theta = \underline{-i\pi}$$

f is analytic inside C_R

$$\therefore \lim_{R \to \infty} \underbrace{\int_{C_R} f(z)\, dz}_{=0} = \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx - i\pi$$

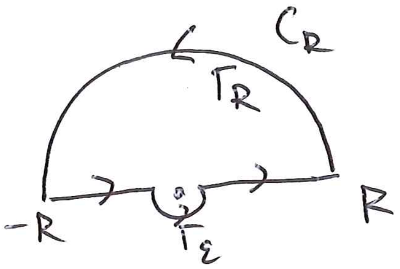
$$\Rightarrow 0 = \int_{-\infty}^{\infty} e^{ix}/x \, dx - \pi i$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \pi i$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \operatorname{Im}(-) = \underline{\pi}.$$

② Proof 2

If we instead try the following contour:



we get

$$\int_{C_R} f = \int_{-R}^{-\varepsilon} f + \int_{\Gamma_\varepsilon} f + \int_{\varepsilon}^{R} f + \int_{\Gamma_R} f$$

these tend to $\int_0^\infty f\,dz$ as $R \to \infty$, $\varepsilon \to 0$

goes to 0 by Jordan's lemma

$$\int_{\Gamma_\varepsilon} f = \int_{-\pi}^{-2\pi} i\left(1 + \varepsilon e^{i\theta} + \frac{(\varepsilon e^{i\theta})^2}{2!} + \dots\right) d\theta$$

$$= \underline{+\pi i}$$

$$\therefore \int_{C_R} f = \int_{-\infty}^{\infty} \frac{e^{iz}}{z}\,dz + \pi i$$

$$\parallel$$

$$2\pi i \operatorname{Res}(f, 0) = 2\pi i \operatorname{Res}\left(\frac{e^{iz}}{z}, z=0\right) = \underline{2\pi i}$$

$$\therefore \quad 2\pi i = \int_{-\infty}^{\infty} \frac{e^{iz}}{z}\,dz + \pi i$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{iz}}{z}\,dz = \pi i$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin z}{z}\,dz = \pi \quad \text{as before.}$$

Addendum:

Here is how to prove that $\int_{\Gamma_R} \frac{e^{iz}}{z}\,dz \to 0$

as $R \to \infty$:

$$z = Re^{i\theta} \qquad dz = Rie^{i\theta}\,d\theta \quad \rightsquigarrow \quad \int_{\Gamma_R} \frac{e^{iz}}{z}\,dz = \int_0^{\pi} \frac{\exp(iRe^{i\theta})}{\cancel{Re^{i\theta}}}\, i\cancel{Re^{i\theta}}\,d\theta$$

$$\left| \int_{\Gamma_R} \frac{e^{iz}}{z}\,dz \right| \leq \int_0^{\pi} \left|\exp(iRe^{i\theta})\right| d\theta$$

$$= \int_0^{\pi} \exp(-R\sin\theta)\,d\theta$$

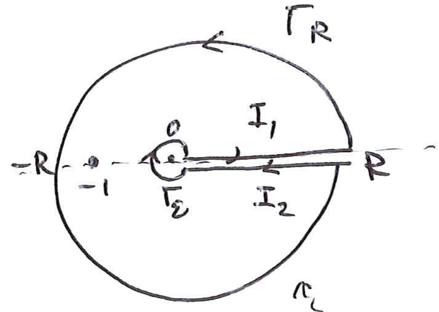
$$\longrightarrow 0 \quad \text{as } R \to \infty$$

since $\sin\theta \geq 0$ for $\theta \in [0,\pi]$.

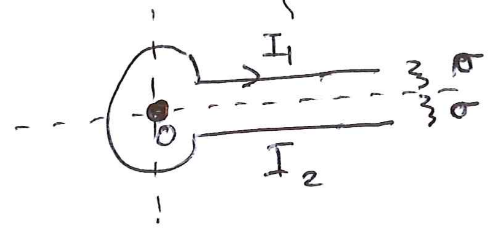
ex1

$$I = \int_0^{\infty} \frac{dx}{\sqrt{x}\,(x+1)} \qquad f(z) = \frac{1}{\sqrt{z}\,(z+1)}$$

Let C_R be the following contour:



We can define a square root along C_R by $\sqrt{z} = \exp\left(\frac{1}{2}\log z\right)$



$$\int_{C_R} = \int_{I_1} + \int_{I_2} + \int_{\Gamma_\varepsilon} + \int_{\Gamma_R}$$

$$\int_{I_1} \frac{dz}{\sqrt{z}\,(z+1)} \longrightarrow \int_0^{\infty} \frac{dz}{\sqrt{z}\,(z+1)} \quad \text{as } R \to \infty \text{ and } \sigma \to 0$$

$$\int_{I_2} f(z) \longrightarrow -\left(\int_{\infty}^{0} \frac{dz}{\sqrt{z}\,(z+1)}\right) = \int_0^{\infty} \frac{dz}{\sqrt{z}\,(z+1)}$$

since $\sqrt{r - \sigma i} = -\sqrt{r} + O(\sigma)$

when $\sigma > 0$

(note that the argument of z on I_2 is close to 2π $\leadsto$ the value of $\sqrt{z}$
$= \exp\left(\frac{1}{2}\log|z| + i \operatorname{Arg} z\right)$
changes by $e^{\pi i} = -1$)

$\int_{\Gamma_R} f \longrightarrow 0$ as $R \to \infty$
(by Jordan)

Need to check $\int_{\Gamma_\varepsilon} \frac{dz}{\sqrt{z}\,(z+1)} = 0$. Then we can find $\int_0^{\infty} f\,dz$ using the residue theorem, since $\int_{C_R} f\,dz = 2\pi i \operatorname{Res}(f, -1)$

$z = \varepsilon e^{i\theta}$

$dz = \qquad \varepsilon i e^{i\theta} d\theta$

$$\int_{-\pi}^{\pi} \frac{\varepsilon i e^{i\theta} d\theta}{\sqrt{\varepsilon i e^{i\theta}}(\varepsilon e^{i\theta}+1)}$$

uniform convergence

$$= \int_{-\pi}^{\pi} \frac{\sqrt{\varepsilon}\, d\theta\, i e^{i\theta}}{\sqrt{i e^{i\theta}}(\varepsilon e^{i\theta}+1)} d\theta \xrightarrow[\varepsilon \to 0]{} \int_{-\pi}^{\pi} 0 = \underline{0}.$$

Hence $\displaystyle\int_{C_R} f = 2\pi i \operatorname{Res}\left(\frac{1}{\sqrt{z}(z+1)}, z = -1\right)$

$$= 2\pi i \left(\frac{1}{i}\right) = \underline{2\pi}$$

$\therefore \displaystyle\int_{I_1} f + \int_{I_2} f = 2\pi$

$\therefore \displaystyle\int_0^\infty \frac{dz}{\sqrt{z}(z+1)} = \frac{1}{2}\left(\lim_{R\to\infty} \int_{I_1} f + \lim_{R\to\infty} \int_{I_2} f\right) = \frac{1}{2}\int_{C_R} f$

$$= \frac{1}{2}(2\pi) = \underline{\pi}.$$

$$\therefore \underline{\int_0^\infty \frac{dz}{\sqrt{z}(z+1)} = \pi}$$