MANDATORY ASSIGNMENT 1

1. $$\frac{\partial u}{\partial x} = 1 + \frac{1\cdot(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} = 1 + \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-x\cdot(2y)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2x(x^2+y^2)^2 - (y^2-x^2)\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}$$

$$= \frac{-2x(x^2+y^2) + 4x(x^2-y^2)}{(x^2+y^2)^3} = \frac{2x^3-6xy^2}{(x^2+y^2)^3}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-2x(x^2+y^2)^2 + 2xy\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4} = \frac{-2x(x^2+y^2)+8xy^2}{(x^2+y^2)^3}$$

$$= \frac{-2x^3+6xy^2}{(x^2+y^2)^3}$$

We see that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, so u is harmonic.

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \frac{2xy}{(x^2+y^2)^2} \Rightarrow v = \frac{-y}{(x^2+y^2)} + g(y)$$

$$\Rightarrow \frac{\partial v}{\partial y} = \frac{-(x^2+y^2)+y(2y)}{(x^2+y^2)^2} + g'(y) = \frac{y^2-x^2}{(x^2+y^2} + g'(y) = \frac{\partial u}{\partial x}$$

This gives $g'(y) = 1$, so $g = y + C$. Then $v(1,0) = 0 + C = 0$, so $C = 0$ and

$$\underline{\underline{v = y - \frac{y}{(x^2+y^2)}}}$$

Then $f = u + iv = (x+iy) + \frac{x-iy}{x^2+y^2} = z + \frac{\bar{z}}{|z|^2} = \underline{\underline{z + \frac{1}{z}}}$.

(One can also insert $x = \frac{1}{2}(z+\bar{z})$, $y = \frac{1}{2i}(z-\bar{z})$)

2. f is of the form $f = \dfrac{z}{cz+d}$

$f(1) = \dfrac{1}{c+d} = 3+3i$, $\quad c+d = \dfrac{1}{3+3i} = \frac{1}{18}(3-3i) = \frac{1}{6} - \frac{1}{6}i$

$f(2) = \dfrac{2}{2c+d} = 3-3i$, $\quad 2c+d = \dfrac{2}{3-3i} = \frac{1}{18}(6+6i) = \frac{1}{3} + \frac{1}{3}i$

$\Rightarrow \quad c = (\frac{1}{3} + \frac{1}{3}i) - (\frac{1}{6} - \frac{1}{6}i) = \frac{1}{6} + \frac{1}{2}i$

$d = \frac{1}{6} - \frac{1}{6}i - (\frac{1}{6} + \frac{1}{2}i) = -\frac{2}{3}i$

$f = \dfrac{z}{(\frac{1}{6}+\frac{1}{2}i)z - \frac{2}{3}i} = \underline{\underline{\dfrac{6z}{(1+3i)z - 4i}}}$

$f(\infty) = \dfrac{6}{1+3i} = \frac{1}{10} \cdot 6(1-3i) = \underline{\underline{0.6 - 1.8i}} =: w_0$

$f(4i) = \dfrac{24i}{4i - 12 - 4i} = \underline{\underline{-2i}}$

$f^{-1}(\infty) = \dfrac{4i}{1+3i} = \frac{1}{10} 4i(1-3i) = \underline{\underline{1.2 + 0.4i}} =: z_0$

$f(\mathbb{R}^*)$ is a circle through the points $0, 3+3i, 3-3i$.
This is the circle $\underline{\underline{|z-3| = 3}}$

$f(i\mathbb{R}^*)$ is a circle through $0, -2i$, orthogonal to $f(\mathbb{R}^*)$. This is the circle $\underline{\underline{|z+i| = 1}}$.

Since the point z_0 is on the line $\{y = \frac{1}{3}x\}$, $f(\{y = \frac{1}{3}x\}^*)$ is a circle through the points $0, 0.6 - 1.8i, \infty$.
This is the line $\underline{\underline{y = -3x \text{ (plus } \infty)}}$.

We have $f^{-1}(z) = \dfrac{-4iz}{-(1+3i)z+6} = \dfrac{4iz}{(1+3i)z-6}$ so $f^{-1}(6) = \dfrac{24i}{18i} = \dfrac{4}{3}$.

So $f^{-1}(\mathbb{R}^*)$ is a circle through the points $0, \frac{4}{3}$, tangent to the imaginary axis. This is the circle $|z - \frac{2}{3}| = \frac{2}{3}$.

$f^{-1}(i\mathbb{R}^*)$ is a circle through $0, 4i$, tangent to the real axis. This is the circle $|z-2i| = 2$.
Sketch on page 5.

3 a) For any $w \in \mathbb{C}$ we must solve $f(z) = \frac{1}{2i}(z - \frac{1}{z}) = w$

This is equivalent to

$$z^2 - 2iwz - 1 = 0$$

which has solutions $z = \dfrac{-2iw \pm \sqrt{-4w^2+4}}{2} = -iw \pm \sqrt{1-w^2}$.

It is clear that the solution is never zero (from the equation)

b) $\frac{1}{2i}(z - \frac{1}{z}) = \frac{1}{2i}(w - \frac{1}{w})$

$$\Updownarrow$$

$$z - \frac{1}{z} = w - \frac{1}{w}$$

$$\Updownarrow$$

$$z - w = \frac{1}{z} - \frac{1}{w} = \frac{w-z}{zw}$$

$\Updownarrow$ when $w \neq z$

$zw = 1$, i.e. $w = \frac{1}{z}$

$f(it) = \frac{1}{2i}(it - \frac{1}{it}) = \frac{1}{2}(t + \frac{1}{t})$

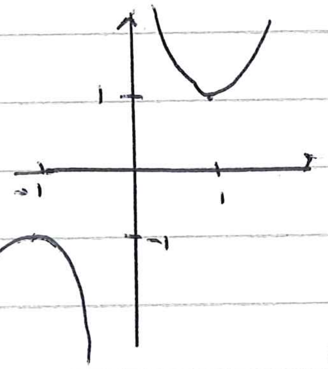


Figure: $\frac{1}{2}(t+\frac{1}{t})$

Every value in $(-\infty,-1)\cup(1,\infty)$ is taken twice. The values ± 1 are taken at $\pm i$.

c) It follows from b) that f takes the same values in the left half plane $\operatorname{Re} z < 0$ as in the right half plane. By a) the values in $(-\infty,-1]\cup[1,\infty)$ are only taken on the imaginary axis. Hence f is an injective map of $H = \{\operatorname{Re} z > 0\}$ onto U. Also $f'(z) = \frac{1}{2i}(1 + \frac{1}{z^2})$ which is zero for $z = \pm i$, hence $f'(z) \neq 0$ in H so f is conformal.

d) We know that $1-w^2$ maps U onto the slit plane $\mathbb{C} \setminus (-\infty, 0]$ so the two roots are $\pm\sqrt{1-w^2}$ where $\sqrt{}$ denotes the principal branch of the square root.
For $w=0$ the solution $iw + \sqrt{1-w^2} = \sqrt{1} = 1 \in H$, so we must use the plus sign.

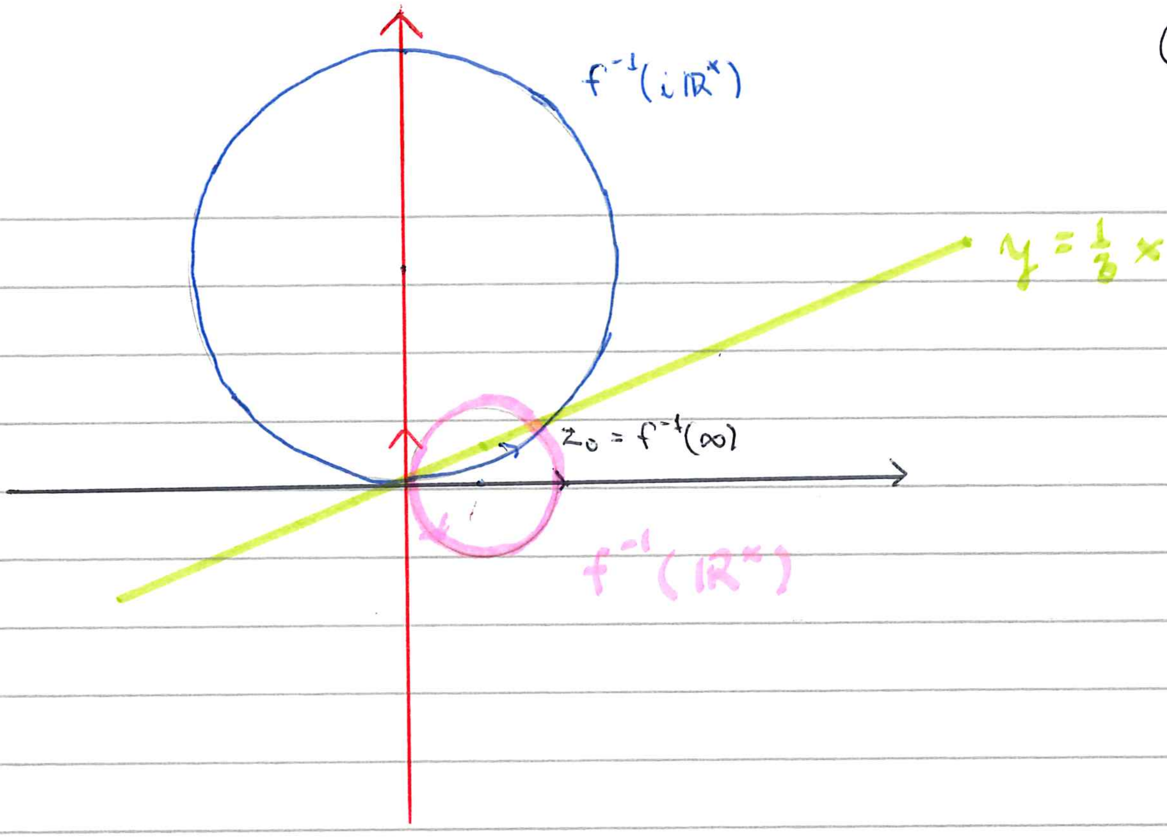
e) $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i} = f(e^{iz})$ is the composition of the conformal mappings e^{iz} and f, hence is conformal.
Since e^{iz} maps D onto H and f maps H onto U, $f(e^{iz})$ maps D onto U.

f) We get $e^{iz} = f^{-1}(w) = iw + \sqrt{1-w^2}$, so
$iz = \operatorname{Log}(iw + \sqrt{1-w^2})$ and

$$z = \sin^{-1}(w) = -i \operatorname{Log}(iw + \sqrt{1-w^2}).$$

$$(\sin^{-1})'(w) = -i \, \frac{i + \frac{1}{2} \cdot \frac{-2w}{\sqrt{1-w^2}}}{iw + \sqrt{1-w^2}}$$

$$= \frac{1 + iw/\sqrt{1-w^2}}{iw + \sqrt{1-w^2}} = \underline{\underline{\frac{1}{\sqrt{1-w^2}}}}$$



$\downarrow f(z)$