MANDATORY ASSIGNMENT NO. 2.

1

a) $\vec{V} = \left(\frac{-2xy}{(x^2+y^2)^2} , \frac{x^2-y^2}{(x^2+y^2)^2} \right) = (P, Q)$

$$\frac{\partial P}{\partial x} = \frac{-2y(x^2+y^2)^2 + 2xy \cdot 2(x^2+y^2) \cdot 2x}{(x^2+y^2)^4}$$

$$= \frac{-2y(x^2+y^2) + 8x^2y}{(x^2+y^2)^3} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$$

By symmetry

$$\frac{\partial P}{\partial y} = \frac{6xy^2 - 2x^3}{(x^2+y^2)^3}$$

$$\frac{\partial Q}{\partial x} = \frac{2x(x^2+y^2)^2 - (x^2-y^2) \cdot 2(x^2+y^2) \cdot 2x}{(x^2+y^2)^4}$$

$$= \frac{2x(x^2+y^2) - 4x^3 + 4xy^2}{(x^2+y^2)^3} = \frac{6xy^2 - 2x^3}{(x^2+y^2)^3}$$

By symmetry

$$\frac{\partial Q}{\partial y} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$$

Hence

$\text{Div } \vec{V} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0$, so the fluid is incompressible

$\text{curl } \vec{V} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$, so the fluid is irrotational.

b) A potential function ϕ (i.e. $\nabla \phi = \vec{V}$) is given by

$$\phi = \int \frac{-2xy}{(x^2+y^2)^2} dx = \frac{y}{(x^2+y^2)} + g(y)$$

$\Rightarrow \frac{\partial \phi}{\partial y} = \frac{(x^2+y^2) - y \cdot 2y}{(x^2+y^2)^2} + g'(y) = \frac{x^2-y^2}{(x^2+y^2)^2} + g'(y)$. We can choose

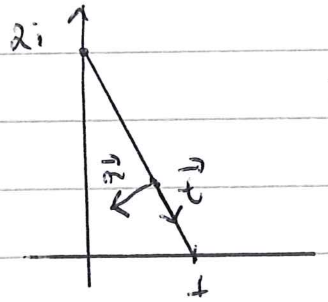
$g = 0$, so $\phi = \frac{y}{(x^2+y^2)}$.

A harmonic conjugate is given by

$$\psi = \frac{x}{(x^2+y^2)}$$ (Integrate CR equations or look at Mandatory assignment no 1).

Hence the complex velocity potential is

$$f = \phi + i\psi = \underline{\underline{\frac{y+ix}{(x^2+y^2)}}} = \frac{i\bar{z}}{|z|^2} = \underline{\underline{\frac{i}{z}}}.$$



If the line γ is directed from $2i$ to 1, the normal vector points from right to left and the flux is

$$\int_\gamma P\,dy - Q\,dx = \int_\gamma d\psi = \psi(1) - \psi(2i) = \frac{1}{1} - \frac{0}{4} = \underline{\underline{1.}}$$

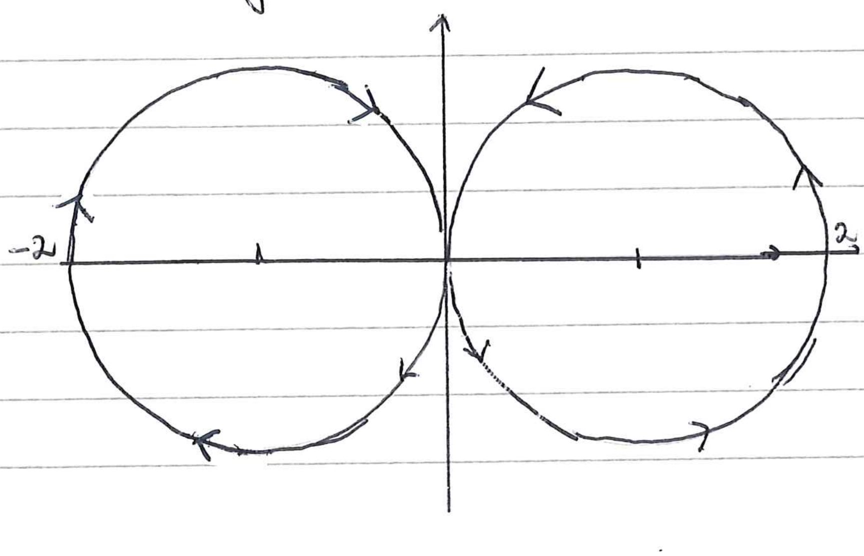
c) The flowlines are given by $\psi(x,y) = \dfrac{x}{x^2+y^2} = C.$

If $C = 0$, this is the y-axis (without the origin).
If $C \neq 0$, this is

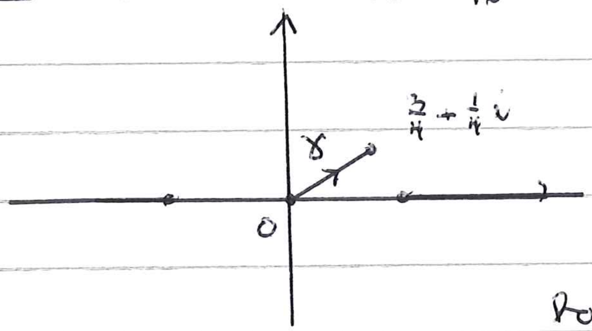
$$x^2 + y^2 = \frac{1}{C}x$$

$$\left(x - \frac{1}{2C}\right)^2 + y^2 = \frac{1}{4C^2},$$

a circle of center $(\frac{1}{2C}, 0)$ and radius $1/2C$.



2. $\left(\frac{3}{4} - \frac{1}{4} i\right)^2 = \frac{9}{16} - \frac{6}{16} i - \frac{1}{16} = \frac{1}{2} - \frac{3}{8} i$



From mandatory assignment no 1 we know that $\frac{1}{\sqrt{1-z^2}}$ has antiderivative

$\sin^{-1}(z) = -i \,\mathrm{Log}(iz + \sqrt{1-z^2})$ in $U = \mathbb{C} \setminus ((-\infty,-1] \cup [1,\infty))$.

Hence

$$\int_\gamma \frac{dz}{\sqrt{1-z^2}} = -i\,\mathrm{Log}\left(iz + \sqrt{1-z^2}\right)\Big|_0^{\frac{3}{4}+\frac{1}{4}i}$$

$$= -i\,\mathrm{Log}\left(\frac{3}{4} i - \frac{1}{4} + \sqrt{1-\left(\frac{1}{2}+\frac{3}{8}i\right)}\right) + i\,\mathrm{Log}\left(0+\sqrt{1}\right)$$

$$= -i\,\mathrm{Log}\left(\frac{3}{4} i - \frac{1}{4} + \sqrt{\frac{1}{2} - \frac{3}{8} i}\right) = -i\,\mathrm{Log}\left(\frac{3}{4} i - \frac{1}{4} + \frac{3}{4} - \frac{1}{4} i\right)$$

$$= -i\,\mathrm{Log}\left(\frac{1}{2} + \frac{1}{2} i\right) = -i\,\mathrm{Log}\left(\frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}}\right) = -i\left(\log 2^{-\frac{1}{2}} + i\frac{\pi}{4}\right)$$

$$= \frac{\pi}{4} + \frac{i}{2}\log 2$$

3 We use Cauchy's integral formula.

a) $$\int_{|z|=1} \frac{dz}{z^2+2z} = \int_{|z|=1} \frac{dz}{z(z+2)} = 2\pi i \left(\frac{1}{z+2}\right)\Big|_{z=0} = \frac{2\pi i}{2} = \pi i$$

b) $$\int_{|z|=2} \frac{\sin z}{z^3 - z^2} dz = \int_{|z|=2} \frac{\sin z}{z^2(z-1)} dz = \int_{|z|=\varepsilon} \frac{\sin z}{z^2(z-1)} dz + \int_{|z-1|=\varepsilon} \frac{\sin z}{z^2(z-1)} dz$$

$$= 2\pi i \frac{d}{dz}\left(\frac{\sin z}{z-1}\right)\Big|_{z=0} + 2\pi i \left(\frac{\sin z}{z^2}\right)\Big|_{z=1} = 2\pi i \left(\frac{(\cos z)(z-1) - \sin z}{(z-1)^2}\right)\Big|_{z=0} + 2\pi i \sin 1$$

$$= -2\pi i + 2\pi i \sin 1 = 2\pi(\sin 1 - 1)\, i$$

4 a) Both the numerator and the denominator has a double zero at $z=0$. Hence the singularity is removable.

b) $1-\cos z \div \sin z^2 =$

$$\frac{1}{2}z^2 - \frac{1}{24}z^4 + \frac{1}{720}z^6 + \mathcal{O}(z^8) \div z^2 - \frac{1}{6}z^6 + \mathcal{O}(z^{10}) = \frac{1}{2} - \frac{1}{24}z^2 + \frac{61}{720}z^4 + \mathcal{O}(z^6)$$

$$-\left(\frac{1}{2}z^2 - \frac{1}{12}z^6 + \mathcal{O}(z^{10})\right)$$

$$-\frac{1}{24}z^4 + \frac{61}{720}z^6 + \mathcal{O}(z^8)$$

$$-\frac{1}{24}z^4 + \mathcal{O}(z^8)$$

$$\frac{61}{720}z^6 + \mathcal{O}(z^8)$$

$$\frac{61}{720}z^6 + \mathcal{O}(z^{10})$$

$$\mathcal{O}(z^8)$$

c) The radius is the distance to the nearest singularity, which is given by

$$z^2 = \pm\pi, \quad z = \begin{matrix}\pm\sqrt{\pi} \\ \pm i\sqrt{\pi}\end{matrix}$$

i.e. $\underline{\underline{\sqrt{\pi}}}$

5. $f(z) = \dfrac{1}{z^3 - z^2} = \dfrac{1}{z^2(z-1)}$

a) f has a double pole at $z=0$, a simple pole at $z=1$ and a triple zero at ∞.

b) We have $\dfrac{1}{z^2(z-1)} = \dfrac{A}{z^2} + \dfrac{B}{z} + \dfrac{C}{z-1}$ We get

$1 = A(z-1) + B(z^2 - z) + Cz^2 = (B+C)z^2 + (A-B)z - A \Rightarrow A = -1,\ B = -1,\ C = 1.$

Hence $f(z) = -\dfrac{1}{z^2} - \dfrac{1}{z} - \displaystyle\sum_{n=0}^{\infty} z^n = -\sum_{n=-2}^{\infty} z^n$ in D_1.

In D_2 we have $f(z) = -\dfrac{1}{z^2} - \dfrac{1}{z} + \displaystyle\sum_{n=-\infty}^{-1} \frac{z^n}{1^{n+1}} = \sum_{n=-\infty}^{-3} z^n$

c) In the annulus $1 < |z+1| < 2$ we have

$$\frac{1}{z} = \sum_{n=-\infty}^{-1} \frac{(z+1)^n}{1^{n+1}} = \sum_{n=-\infty}^{-1} (z+1)^n$$

$$\frac{1}{z^2} = -\frac{d}{dz}\left(\frac{1}{z}\right) = -\sum_{n=-\infty}^{-1} n(z+1)^{n-1} = -\sum_{n=-\infty}^{-2} (n+1)(z+1)^n$$

$$-\frac{1}{z^2} - \frac{1}{z} = \sum_{n=-\infty}^{-2} (n+1-1)(z+1)^n - (z+1)^{-1} = \sum_{n=-\infty}^{-1} n(z+1)^n$$

$$\frac{1}{z-1} = -\sum_{n=0}^{\infty} \frac{(z+1)^n}{(1-(-1))^{n+1}} = -\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} (z+1)^n$$

Hence the Laurent series is $f(z) = \displaystyle\sum_{n=-\infty}^{\infty} a_n (z+1)^n$ where

$$a_n = \begin{cases} n & \text{if } n < 0 \\ -\dfrac{1}{2^{n+1}} & \text{if } n \geq 0 \end{cases}$$