

VI. 1

a) $f(z) = \frac{1}{z^2 - z} = \frac{1}{z(z-1)}$. The function is analytic in $0 < |z| < 1$ and $|z| > 1$.

The Laurent expansion for $|z| > 1$ is

$$f(z) = \frac{1}{z^2} \left(\frac{1}{1 - 1/z} \right) \underset{|1/z| < 1}{=} \frac{1}{z^2} \sum_{h=0}^{\infty} z^{-h} = \sum_{h=0}^{\infty} z^{-h-2}$$

(the Laurent expansion "at ∞ ").

For $0 < |z| < 1$ we write

$$\begin{aligned} f(z) &= \frac{1}{z-1} - \frac{1}{z} = - \sum_{h=0}^{\infty} z^h - \frac{1}{z} \\ &= - \sum_{h=-1}^{\infty} z^h. \end{aligned}$$

b) $f(z) = \frac{z-1}{z+1}$ is analytic on $|z| < 1$ and $|z| > 1$.

So the Laurent expansions are just the Taylor series at 0 and ∞ respectively. I.e.

$$\text{On } |z| < 1: f(z) = (z-1) \sum_{h=0}^{\infty} (-1)^h z^h = -1 - 2 \sum_{h=1}^{\infty} (-1)^h z^h$$

On $|z| > 1$:

$$\begin{aligned} f(z) &= \frac{z-1}{z} \left(\frac{1}{1 - (-1/z)} \right) \\ &= \left(1 - \frac{1}{z} \right) \sum_{h=0}^{\infty} (-1)^h z^{-h} = 1 + 2 \sum_{h=-\infty}^{-1} (-1)^h z^h \end{aligned}$$

V1.1 c) $f(z) = \frac{1}{(z^2-1)(z^2-4)}$ is analytic on

$$|z| < 1, \quad 1 < |z| < 2 \quad \text{and} \quad |z| > 2.$$

$|z| < 1$: Taylor expansion.

$$\frac{1}{z^2-1} = -\sum_{h=0}^{\infty} z^{2h}, \quad \frac{1}{z^2-4} = -\frac{1}{4} \sum_{h=0}^{\infty} \frac{z^{2h}}{4^h}$$

$$\begin{aligned} \text{So } f(z) &= \frac{1}{4} \left(\sum_{h=0}^{\infty} z^{2h} \right) \left(\sum_{h=0}^{\infty} \frac{z^{2h}}{4^h} \right) \\ &= \frac{1}{4} \sum_{m=0}^{\infty} \left(\sum_{h=0}^m \frac{1}{4^h} \right) z^{2m} \\ &= \frac{1}{4} \sum_{m=0}^{\infty} \left(\frac{1 - 4^{-m-1}}{1 - 4^{-1}} \right) z^{2m} \\ &= \frac{1}{4} \sum_{m=0}^{\infty} \frac{4 - 4^{-m}}{3} z^{2m} = \frac{1}{12} \sum_{m=0}^{\infty} (4 - 4^{-m}) z^{2m}. \end{aligned}$$

$$1 < |z| < 2: \quad \frac{1}{z^2-1} = \frac{1}{z^2} \left(\frac{1}{1-1/z^2} \right) = \frac{1}{z^2} \sum_{h=0}^{\infty} z^{-2h}$$

$$\text{So } f(z) = -\frac{1}{4} z^2 \left(\sum_{h=-\infty}^0 z^{2h} \right) \left(\sum_{h=0}^{\infty} \frac{z^{2h}}{4^h} \right) = \sum_{h=-\infty}^{\infty} a_m z^m,$$

where $a_m = \sum_{\substack{h+l=m \\ h \geq 0 \\ l \leq 0}} \frac{1}{4^h}$. Note that if $m \geq 0$

$$a_m = \sum_{h=m}^{\infty} \frac{1}{4^h} = \frac{1}{1-4^{-1}} - \frac{1-4^{-m}}{1-4^{-1}} = \frac{4^{-m+1}}{3}.$$

Moreover, if $m < 0$, then

$$a_m = \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{4}{3}.$$

In conclusion

$$\begin{aligned} f(z) &= \sum_{m=-\infty}^{\infty} a_m z^m = -\frac{1}{4z^2} \left(\frac{4}{3} \sum_{m=-\infty}^{-1} z^{2m} + \frac{4}{3} \sum_{m=0}^{\infty} \frac{z^{2m}}{4^m} \right) \\ &= -\frac{1}{3} \sum_{m=-\infty}^{-2} z^{2m} - \frac{1}{3} \sum_{m=0}^{\infty} \frac{z^{2m-2}}{4^m} \\ &= -\frac{1}{3} \sum_{m=-\infty}^{-2} z^{2m} - \frac{z^{-2}}{3} - \frac{1}{12} \sum_{m=0}^{\infty} \frac{z^{2m}}{4^m} \\ &= -\frac{1}{3} \sum_{m=-\infty}^{-1} z^{2m} - \frac{1}{12} \sum_{m=0}^{\infty} \frac{z^{2m}}{4^m}. \end{aligned}$$

$|z| > 2$: similarity

VI

a) $f(z) = \frac{1}{z(z-1)}$ is analytic on $|z+1| < 1$,

$1 < |z+1| < 2$ and $|z+1| > 2$.

The Laurent expansion converging at $z = \frac{1}{2}$ is then the expansion for f in $1 < |z+1| < 2$.

We have

$$f(z) = \frac{1}{z-1} - \frac{1}{z}$$

$$= -\frac{1}{2} \frac{1}{1 - \frac{z+1}{2}} + \frac{1}{z+1} \frac{1}{1 - 1/(z+1)}$$

$$= -\frac{1}{2} \sum_{h=0}^{\infty} 2^{-h} (z+1)^h + \sum_{h=0}^{\infty} \frac{1}{(z+1)^{h+1}} = \sum_{h=-\infty}^{-1} (z+1)^h + \sum_{h=0}^{\infty} \frac{(z+1)^h}{2^{h+1}}$$



