

IV.2.2. A primitive for $1/z$ on $\mathbb{C} \setminus (-\infty, 0]$ is $\text{Log}(z)$ (principal branch). The fundamental theorem of calculus says that

$$\int_{\gamma} 1/z \, dz = \text{Log}(\pi i) - \text{Log}(-\pi i) = i\pi$$

as long as γ is in $\mathbb{C} \setminus (-\infty, 0]$.

If we instead take a path γ in the left half plane, the principal branch of $\log(z)$ is not a primitive for $1/z$ where γ is. So we take instead the branch

$$\text{Log}(ve^{i\theta}) = \ln(v) + i\theta, \quad 0 < \theta \leq 2\pi.$$

Then

$$\begin{aligned} \int_{\gamma} 1/z \, dz &= \text{Log}(\pi i) - \text{Log}(-\pi i) \\ &= \ln(\pi) - \ln(\pi) + i\left(\frac{\pi}{2} - \frac{3\pi}{2}\right) = -i\pi. \end{aligned}$$

IV.2.5. $\Rightarrow$: If f has a primitive in D , call it F ,

then $\int_{\gamma} f(z) \, dz = F(B) - F(A)$ for any path from

A to B . γ closed $(\Rightarrow A=B)$, so $\int_{\gamma} f(z) \, dz = 0$.

$\Leftarrow$: Recall that a differential $Pdx + Qdy$ is

exact iff $\int_{\gamma_1} Pdx + Qdy = \int_{\gamma_2} Pdx + Qdy$ for any

two paths γ_1, γ_2 with the same endpoints. We have

$$0 = \int_{\gamma_1 - \gamma_2} f(z) \, dz \Rightarrow \int_{\gamma_1} f(z) \, dz = \int_{\gamma_2} f(z) \, dz, \text{ so}$$

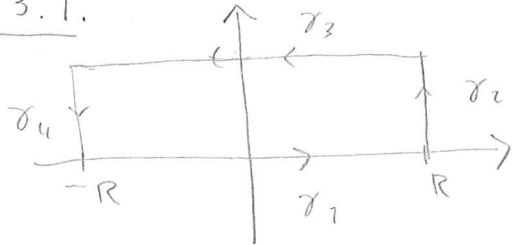
the differential $f(z) \, dz = (u + iv)dx + (-v + iu)dy$ is exact.

IV.2.5 cont. I.e. there is an analytic function F

s.t. $dF = (u+iv)dx + (-v+iu)dy$. Thus

$$F'(z) = \frac{\partial F}{\partial x} = u+iv = f(z).$$

IV.3.1.



Because $e^{-z^2/2}$ is analytic

$$\int_{\gamma} e^{-z^2/2} dz = 0.$$

Google "Gaussian integral"

$$\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4. \quad \int_{-\infty}^{\infty} e^{-x^2/2} dx \stackrel{\downarrow}{=} \sqrt{2\pi}$$

$$I_1 = \int_{\gamma_2} e^{-z^2/2} dz = - \int_{-R}^R e^{-(-s+it)^2/2} ds = -e^{t^2/2} \int_{-R}^R e^{-s^2/2} e^{-its} ds$$

$$I_2 = \left(\int_{\gamma_1} + \int_{\gamma_3} \right) e^{-z^2/2} dz = e^{-R^2/2} \int_0^t e^{s^2/2} (e^{-iRs} - e^{iRs}) ds$$

Thus, from (1) $0 = \int_{\gamma} e^{-z^2/2} dz = I_1 + I_2 + \int_{-R}^R e^{-x^2/2} dx$

we obtain as $R \rightarrow \infty$

$$e^{t^2/2} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-itx} dx = \sqrt{2\pi}$$

because $\lim_{R \rightarrow \infty} I_2 = 0$. (The integral has the bound

$$|I_2| \leq e^{-R^2/2} t e^{t^2/2} \rightarrow 0 \quad \text{as } R \rightarrow \infty$$

IV.3.5 Because f is analytic we have $\int_{\partial D} f dz = 0$.

Hence

$$2i \text{Area}(D) = \int_{\partial D} \bar{z} dz = \int_{\partial D} (\bar{z} - f(z)) dz.$$

Exercise IV.1.4

(Use Green's theorem)

ML-estimate:

$$2 \text{Area} = \left| \int_{\partial D} \bar{z} - f(z) dz \right| \leq \sup_{\partial D} |\bar{z} - f(z)| \cdot \text{length}(\partial D).$$

This gives the wanted formula. An example with equality: $f = 0$, $D = \{z \mid |z| < 1\}$. Then

$$1 = \sup_{|z|=1} |\bar{z}| = \sup_{|z|=1} |\bar{z} - f(z)|$$

$$\text{and } \frac{2 \text{Area}(D)}{\text{length}(\partial D)} = \frac{2\pi}{2\pi} = 1.$$

IV.4.1 (some of them)

The point is to use Cauchy's integral formula (CIF).

$$a) \int_{|z|=2} \frac{z^n}{z-1} = 2\pi i, \text{ because with } f(z) = z^n$$

$$\text{we get from CIF that } 1 = f(1) = \frac{1}{2\pi i} \int_{|z|=2} \frac{f(z)}{z-1} dz.$$

$$g) \int_{|z|=1} \frac{dz}{z^2(z^2-4)e^z} = \int_{|z|=1} \frac{f(z)}{(z-0)^2} dz \text{ where } f(z) = \frac{\bar{e}^z}{z^2-4}$$

IV.4.1 d) cont. We have

$$\textcircled{1} \quad f'(z) = \frac{-e^{-z}(z^2-4) - 2ze^{-z}}{(z^2-4)^2} = -\frac{e^{-z}(z^2+2z-4)}{z^2-4}$$

$$f'(0) = -1.$$

Moreover

$$f'(0) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{(z-0)^2} dz$$

$$\textcircled{2} \quad \Rightarrow \int_{|z|=1} \frac{dz}{z^2(z^2-4)e^z} = -2\pi i.$$

(IV.4.1h is on the next page)

IV.4.4. By CIF we have

$$|f(z_0)| = \frac{1}{2\pi} \left| \int_{\partial D} \frac{f(z)}{z-z_0} dz \right|$$

$$\leq \frac{1}{2\pi} \text{length}(\partial D) \cdot \sup_{\partial D} \left| \frac{f(z)}{z-z_0} \right|$$

$$\leq \underbrace{\frac{\text{length}(\partial D)}{2\pi} \cdot \sup_{\partial D} \left| \frac{1}{z-z_0} \right|}_{C} \sup_{\partial D} |f(z)|$$

$$\text{We also get } |f(z_0)^n| \leq C \cdot \sup_{\partial D} |f(z)|^n$$

$$\Leftrightarrow |f(z_0)|^n \leq C \cdot \sup_{\partial D} |f(z)|^n$$

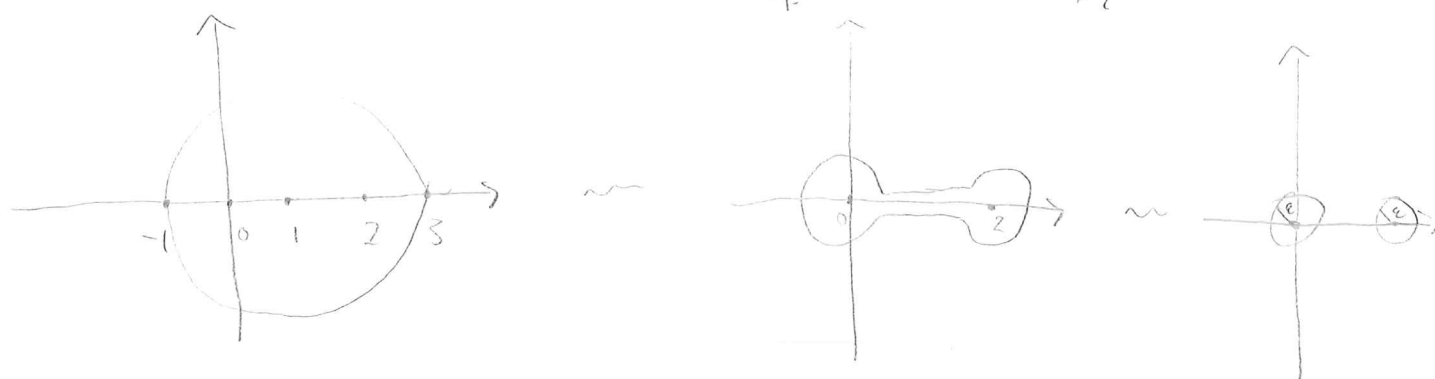
⊃ Taking n 'th root we get

$$|f(z_0)| \leq C^{1/n} \sup_{\partial D} |f(z)| \text{ for all } n, \quad C^{1/n} \rightarrow 1.$$

IV.4.1h Note that the circle $|z-1|=2$ contains 0 and 2. Hence, we can not use the CLF on either $\frac{1}{(z^2-4)e^z}$ or $\frac{1}{z(z+2)e^z}$ (because they are not analytic on the disc $|z-1|\leq 2$).

However, the integral $\int_{|z-1|=2} f(z) dz$ is the same as

the sum of the integrals $\int_{D_1} f(z) dz + \int_{D_2} f(z) dz :$



$$\begin{aligned}
 \text{So} \quad \int_{|z-1|=2} \frac{dz}{z(z^2-4)e^z} &= \int_{|z|= \epsilon} \frac{1}{z-0} \left\{ \frac{1}{(z^2-4)e^z} \right\} g(z) + \int_{|z-2|= \epsilon} \frac{1}{z-2} \left\{ \frac{1}{z(z+2)e^z} \right\} h(z) dz \\
 &= 2\pi i \cdot g(0) + 2\pi i \cdot h(2) \\
 &= 2\pi i \cdot \left(-\frac{1}{4}\right) + 2\pi i \cdot \frac{e^{-2}}{8} \\
 &= \frac{\pi i}{2} \left(\frac{1}{2} e^{-2} - 1 \right).
 \end{aligned}$$