

II.3-II.5: CAUCHY–RIEMANN, INVERSE FUNCTIONS & HARMONIC FUNCTIONS

Recall ([Gamelin, p. 42]) the definition of what it means for a complex function $f(z)$ is **complex differentiable** ($\mathbb{C}$ -differentiable):

$$f(z) \text{ is } \mathbb{C}\text{-differentiable at } z_0 \text{ if } \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists .}$$

Definition ([Gamelin, p. 45]). We say that $f(z)$ is **analytic** (= **holomorphic**) on a domain D if $f(z)$ is $\mathbb{C}$ -differentiable at every point of D , and $f'(z)$ is a continuous function.¹

1. CAUCHY–RIEMANN EQUATIONS

Proposition ([Gamelin, p. 47]). *A complex function f on a domain D is analytic if and only if, writing*

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

*we have that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and are continuous, and moreover these satisfy the **Cauchy–Riemann equations***

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned}$$

Example ([Gamelin p. 49]). Consider $f(z) = e^z = e^x(\cos y + i \sin y)$, and let us show that this is $\mathbb{C}$ -differentiable. We have $u = e^x \cos y$, $v = e^x \sin y$, and find

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos y = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -e^x \sin y = -\frac{\partial v}{\partial x}, \end{aligned}$$

so e^z is analytic on $D = \mathbb{C}$.

Proposition ([Gamelin p. 50]). *If $f(z)$ is an analytic function on a domain D such that $f(z)$ is real for all $z \in D$, then $f(z)$ is constant.*

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¹It turns out that the condition “ $f'(z)$ is continuous” is automatic (Goursat’s theorem), see [Gam, p. 123].

Proof. We have $f(z) = u(x, y) + iv(x, y)$ with $v(x, y) = 0$. The CR equations therefore give

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} = 0 \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} = 0.\end{aligned}$$

Now since D is a domain, this means that $u(x, y)$ is constant ([Gamelin p. 38]), and therefore $f(z) = u(x, y)$ is constant. $\square$

1.1. Constructing analytic functions. The following results say that the usual ways of constructing new functions from old ones preserve the notion of being complex differentiable.

Proposition ([Gamelin p. 43]). *Let f, g be analytic functions on a domain D , then $f + g$, fg and f/g (assuming $g(z) \neq 0$ for all $z \in D$) are analytic, and we have*

$$\begin{aligned}(f + g)'(z) &= f'(z) + g'(z) \\ (fg)'(z) &= f'(z)g(z) + f(z)g'(z) \\ (f/g)'(z) &= \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}\end{aligned}$$

Proposition (The chain rule ([Gamelin p. 44]).) *Let f be analytic on a domain D , and let g be analytic on a domain E . If $f(z) \in E$ for all $z \in D$, then $g \circ f$ is analytic on D , and we have*

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

Example (Oblig 1, 2021). Prove that $f(z) = e^{1/z}$ is analytic on $D = \mathbb{C} \setminus \{0\}$ and compute $f'(z)$.

We know that $1/z$ is analytic and e^z is analytic, so the chain rule gives $e^{1/z}$ is analytic, and moreover

$$(e^{1/z})' = e^{1/z} \cdot \frac{-1}{z^2}.$$

Alternatively write $e^{1/z}$ in terms of real and imaginary parts:

$$e^{1/z} = e^{\frac{x-iy}{x^2+y^2}} = e^{\frac{x}{x^2+y^2}} \left(\cos\left(\frac{-y}{x^2+y^2}\right) + i \sin\left(\frac{-y}{x^2+y^2}\right) \right),$$

hence

$$e^{1/z} = u(x, y) + iv(x, y)$$

with

$$\begin{aligned}u(x, y) &= e^{\frac{x}{x^2+y^2}} \cos\left(\frac{-y}{x^2+y^2}\right) \\ v(x, y) &= e^{\frac{x}{x^2+y^2}} \sin\left(\frac{-y}{x^2+y^2}\right).\end{aligned}$$

Now one can check the Cauchy–Riemann equations for u and v and compute $f'(z)$ via

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + i \frac{\partial v}{\partial x}(x, y).$$

That's clearly a lot of work, so consider this an advertisement for the chain rule.

1.2. The inverse function theorem.

Proposition (The inverse function theorem, [Gamelin p. 51]). *Let $z_0 \in D \subseteq \mathbb{C}$ be a point in a domain, and let $f(z)$ be analytic on D . If $f'(z_0) \neq 0$, then there exists an open $U \ni z_0$ such that $f(z)$ is one-to-one on U , the image $V = f(U)$ is open, and the inverse function f^{-1} is analytic on V with*

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}$$

for all $w \in V$.

Proof. Consider the Jacobian matrix

$$J(f)(z) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix},$$

where the second equality uses the Cauchy–Riemann equations. This has determinant

$$|J(f)(z)| = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 = \left| \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right|^2 = |f'(z)|^2.$$

So if $f'(z_0) \neq 0$, the matrix $J(f)(z_0)$ is invertible.

The inverse function theorem from multivariable calculus then says that we can find a open sets $U \ni z_0$ and $V = f(U)$ and an inverse $f^{-1} = a(x, y) + ib(x, y)$ on V where a and b have continuous partial derivatives.

We need to check that f^{-1} is analytic, which means that a and b satisfy the Cauchy–Riemann equations. We have

$$\begin{aligned} \begin{pmatrix} \frac{\partial a}{\partial x}(f(z_0)) & \frac{\partial a}{\partial y}(f(z_0)) \\ \frac{\partial b}{\partial x}(f(z_0)) & \frac{\partial b}{\partial y}(f(z_0)) \end{pmatrix} &= J(f^{-1})(f(z_0)) = J(f)(z_0)^{-1} \\ &= \frac{1}{\det(J(f)(z_0))} \begin{pmatrix} \frac{\partial u}{\partial x}(z_0) & \frac{\partial v}{\partial x}(z_0) \\ -\frac{\partial v}{\partial x}(z_0) & \frac{\partial u}{\partial x}(z_0) \end{pmatrix}. \end{aligned}$$

From this we see that a and b satisfy the Cauchy–Riemann equation.

Since $(f \circ f^{-1})(w) = w$, the chain rule gives

$$f'(f^{-1}(w))(f^{-1})'(w) = 1,$$

which in turn gives

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}.$$

□

Example. Let $f(z) = e^z$, then for every point z_0 not of the form $z_0 = x$ with x real and negative, the function $\text{Log}(w)$ is an inverse to f , and computing shows that

$$\text{Log}(w)' = \frac{1}{e^{\text{Log}(w)}} = \frac{1}{w}.$$

2. II.5 – HARMONIC FUNCTIONS

So far we have given a set of differential equations for complex-valued functions and claimed that the functions satisfying these are $\mathbb{C}$ -differentiable. What can we then further say about their real and imaginary parts, i.e. the functions u and v ?

Definition ([Gamelin p. 54]). For real functions in n variables, the **Laplacian** Δ is the operator such that

$$\Delta(f(x_1, \dots, x_n)) = \sum_{i=1}^n \frac{\partial^2 f}{(\partial x_i)^2}(x_1, \dots, x_n).$$

A function f is **harmonic** if it has continuous second order derivatives, and satisfies $\Delta(f) = 0$, in other words if it satisfies **Laplace's equation**

$$\sum_{i=1}^n \frac{\partial^2 f}{(\partial x_i)^2} = 0.$$

Example. Simple examples of harmonic functions are: Constant functions, linear functions, $f(x, y) = x^2 - y^2$, $f(x, y) = xy$, $f(x, y) = e^x \cos y$, $f(x, y) = \log(x^2 + y^2) \dots$

Theorem (Gamelin p. 55). *If the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic and u and v have continuous second-order derivatives², then u and v are harmonic.*

Proof. Using the Cauchy–Riemann equations, we can compute

$$\begin{aligned} \Delta(u) &= \frac{\partial^2 u}{(\partial x)^2} + \frac{\partial^2 u}{(\partial y)^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) \\ &\stackrel{\text{CR eqns}}{=} \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial y} \left(-\frac{\partial v}{\partial x} \right) = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0. \end{aligned}$$

In the last step we exchange the order of differentiation, which we are allowed to do since v has continuous second-order partial derivatives. $\square$

Now suppose that we are given a harmonic function $u(x, y)$ on a domain D . **Question:** Can we find an analytic function $f(z)$ such that $f(z) = u + iv$?

Definition (Gamelin p.55). If the real function $u(x, y)$ is harmonic on a domain D , and $v(x, y)$ is a function such that $u + iv$ is analytic, then v is a **harmonic conjugate** of u .

We first observe that *if* such a v exists, then it is uniquely determined up to a constant.

²The condition on second-order derivatives is actually automatic, see [Gamelin p. 115].

Lemma. Let $u(x, y)$ be a function defined on a domain D , and let v_1, v_2 be harmonic conjugates of u . Then $v_1 = v_2 + C$ for some real constant C .

Proof. Let $f_1(z) = u + iv_1, f_2(z) = u + iv_2$. The function

$$f_1(z) - f_2(z) = i(v_1(z) - v_2(z))$$

is analytic, hence

$$g(z) = v_1(z) - v_2(z)$$

is analytic. But $g(z)$ is real-valued and D is a domain, so $g(z) = C$ for some constant C , as we saw last lecture. $\square$

Theorem. If $u(x, y)$ is harmonic on an open disc D , then there exists a harmonic conjugate v of u on D .

Proof. The function v is a harmonic conjugate of u if and only if the Cauchy–Riemann equations

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}$$

hold. We are given u as input data, and try to find a v so that these equations hold.

We pick a point $(x_0, y_0) \in D$, and define v in steps as follows.

Step 0: Since v is only defined up to a constant anyway, we set $v(x_0, y_0) = 0$.

Step 1: For $(x, y_0) \in D$, if the CR equations hold, we must have

$$v(x, y_0) - v(x_0, y_0) = \int_{x_0}^x \frac{\partial v}{\partial x}(t, y_0) dt = \int_{x_0}^x -\frac{\partial u}{\partial y}(t, y_0) dt,$$

so we define

$$v(x, y_0) = \int_{x_0}^x -\frac{\partial u}{\partial y}(t, y_0) dt.$$

Step 2: For a general point $(x, y) \in D$, if the CR equations hold we have

$$v(x, y) - v(x, y_0) = \int_{y_0}^y \frac{\partial v}{\partial y}(x, t) dt = \int_{y_0}^y \frac{\partial u}{\partial x}(x, t) dt,$$

so we define

$$v(x, y) = v(x, y_0) + \int_{y_0}^y \frac{\partial u}{\partial x}(x, t) dt = \int_{x_0}^x -\frac{\partial u}{\partial y}(t, y_0) dt + \int_{y_0}^y \frac{\partial u}{\partial x}(x, t) dt$$

Now we've defined v , and it remains to check that the Cauchy–Riemann equations hold:

$$(2.1) \quad \frac{\partial v}{\partial x}(x, y) = -\frac{\partial u}{\partial y}(x, y_0) + \int_{y_0}^y \frac{\partial^2 u}{(\partial x)^2}(x, t) dt$$

Now we can use that u is harmonic, so $\frac{\partial^2 u}{(\partial x)^2} = -\frac{\partial^2 u}{(\partial y)^2}$. Therefore

$$\int_{y_0}^y \frac{\partial^2 u}{(\partial x)^2}(x, t) dt = - \int_{y_0}^y \frac{\partial^2 u}{(\partial y)^2}(x, t) dt = -\frac{\partial u}{\partial y}(x, y) + \frac{\partial u}{\partial y}(x, y_0).$$

Inserting in (2.1) shows that the CR equation $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ holds.

We leave the computation of $\frac{\partial v}{\partial y}$ as an exercise. □

Example. The above proof works whenever D has a particular shape so that we can reach any point from (x_0, y_0) . by first going horizontally, then vertically, e.g. a rectangle with horizontal and vertical edges.

As we'll see, the statement holds more generally, essentially when D has no “holes”, the technical term is “simply connected”.

The statement can fail for general D : The function $u = \log(\sqrt{x^2 + y^2})$ is the real part of (every) logarithm function, and is harmonic on $D = \mathbb{C} \setminus \{0\}$, but has no harmonic conjugate on D .