

II.6-II.7: CONFORMAL MAPS AND FRACTIONAL LINEAR TRANSFORMATIONS

Check out <http://davidbau.com/conformal/> for a tool to visualise analytic functions as conformal maps.

ANALYTIC FUNCTIONS AND CONFORMAL MAPPINGS

Question: Given an analytic function $f(z)$ and a set $E \subseteq \mathbb{C}$, what does the image $f(E)$ look like? What geometric properties does the mapping $z \mapsto f(z)$ have?

Let's look at what $f(z)$ does to vectors. Given a path $\gamma(t) = x(t) + iy(t)$, $0 \leq t \leq 1$ in $\mathbb{C}$, its **tangent vector** at $\gamma(0)$ is the vector $\gamma'(0) = \lim_{t \rightarrow 0} \frac{\gamma(t) - \gamma(0)}{t}$.

Lemma. *If $f(z)$ is analytic and $\gamma(t)$, $0 \leq t \leq 1$ is a smooth path, then the tangent vector of the path $(f \circ \gamma)(t)$ at $f(\gamma(0))$ is*

$$f'(\gamma(0))\gamma'(0).$$

Proof. This is essentially the chain rule: If $\gamma'(t) \neq 0$, then for $0 < t \ll 1$ we have $\gamma(t) \neq \gamma(0)$, so

$$(f \circ \gamma)'(0) = \lim_{t \rightarrow 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t} = \lim_{t \rightarrow 0} \frac{f(\gamma(t)) - f(\gamma(0))}{\gamma(t) - \gamma(0)} \frac{\gamma(t) - \gamma(0)}{t} = f'(\gamma(0))\gamma'(0).$$

If $\gamma'(0) = 0$, an extra argument is needed (which we skip). $\square$

We picture the transformation $\gamma'(0) \mapsto (f \circ \gamma)'(0)$ by drawing the vector $\gamma'(0)$ with its tail in $\gamma(0)$, and transforming it to the vector $f'(\gamma(0))\gamma'(0)$, drawn with its tail in $f(\gamma(0))$.

Definition. A complex function $f(z)$ is **conformal at z_0** if 1 and 2 below hold:

- (1) For every path $\gamma(t)$, $0 \leq t \leq 1$ with $\gamma(0) = z_0$ and $\gamma'(0) \neq 0$, we have $(f \circ \gamma)'(0) \neq 0$.
- (2) For every pair of paths $\gamma_0(t)$, $\gamma_1(t)$, $0 \leq t \leq 1$ such that $z_0 = \gamma_0(0) = \gamma_1(0)$, then the angle between $(f \circ \gamma_0)'(0)$ and $(f \circ \gamma_1)'(0)$ equals that between $\gamma_0'(0)$ and $\gamma_1'(0)$ *with orientation*.

Example. For any $a \in \mathbb{C}$, the function $f(z) = z + b$ is conformal.

If $a \neq 0$, then the function $g(z) = az$ is conformal.

The function $f(z) = \bar{z}$ is *not* conformal, because angles are reversed.

Theorem. *Let $f(z)$ be analytic at z_0 with $f'(z_0) \neq 0$. Then $f(z)$ is conformal at z_0 .*

Proof. If γ_0, γ_1 are paths with $z_0 = \gamma_0(0) = \gamma_1(0)$, then we have

$$(f \circ \gamma_0)'(0) = f'(z_0)\gamma_0'(0) \quad (f \circ \gamma_1)'(0) = f'(z_0)\gamma_1'(0),$$

and if $f'(z_0) \neq 0$, then the angle between $f'(z_0)\gamma_0'(0)$ and $f'(z_0)\gamma_1'(0)$ is the same as that between $\gamma_0'(0)$ and $\gamma_1'(0)$. $\square$

FRACTIONAL LINEAR TRANSFORMATIONS

Definition. A **fractional linear transformation** is a function of the form $f(z) = \frac{az+b}{cz+d}$, where $ad - bc \neq 0$.

Note. The presentation of $f(z)$ in this way is not quite unique, since we can always rescale the coefficients a, b, c, d . For any $0 \neq \lambda \in \mathbb{C}$, we have

$$\frac{az+b}{cz+d} = \frac{\lambda az + \lambda b}{\lambda cz + \lambda d}.$$

Let $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. The function $f(z)$ is analytic for all z such that $cz + d \neq 0$. In the point where $cz + d = 0$ (meaning $z = -d/c$ when $c \neq 0$), we define $f(-d/c) = \infty$. We define

$$f(\infty) = \lim_{z \rightarrow \infty} \frac{az+b}{cz+d} = \begin{cases} \frac{a}{c} & \text{if } c \neq 0 \\ \infty & \text{if } c = 0 \end{cases}.$$

So in particular a fractional linear transformation can be thought of as a map from $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$.

Composing fractional linear transformations. Fractional linear transformations compose nicely.

Theorem. If $f_1(z) = \frac{a_1z+b_1}{c_1z+d_1}$ and $f_2(z) = \frac{a_2z+b_2}{c_2z+d_2}$, then

$$(f_1 \circ f_2)(z) = \frac{az+b}{cz+d},$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Proof. A dirty computation shows

$$\frac{a_1 \frac{a_2z+b_2}{c_2z+d_2} + b_1}{c_1 \frac{a_2z+b_2}{c_2z+d_2} + d_1} = \frac{(a_1a_2 + b_1c_2)z + (a_1b_2 + b_1d_2)}{(c_1a_2 + d_1c_2)z + (c_1b_2 + d_1d_2)}.$$

$\square$

By our definition, in a fractional linear transformation $f(z) = \frac{az+b}{cz+d}$, we must have $ad - bc \neq 0$. This means that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible.

Theorem. The function $f(z) = \frac{az+b}{cz+d}$ has an inverse $g(z)$ given by $g(z) = \frac{dz-b}{-cz+a}$.

Proof. Since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix},$$

we have

$$f(g(z)) = \frac{(ad - bc)z + 0}{0z + (ad - bc)} = z.$$

□

In other words, a FLT defines a bijection from $\mathbb{C}^*$ to $\mathbb{C}^*$.

The geometry of fractional linear transformations. There are three especially simple classes of fractional linear transformations. A **dilation** is a function $f(z) = az$, a **translation** is a function $f(z) = z + b$, and an **inversion** is the function $f(z) = \frac{1}{z}$.

Theorem. *Any fractional linear transformation can be written as a composition of dilations, translations and inversions.*

Proof. Let $f(z) = \frac{az+b}{cz+d}$, and assume $c \neq 0$ (the case $c = 0$ is easier). We can obtain $f(z)$ as the composition of the following sequence of transformations

$$\begin{aligned} z &\mapsto cz, & \text{Composition} &= z \mapsto cz \\ z &\mapsto z + d, & \text{Composition} &= z \mapsto cz + d \\ z &\mapsto z^{-1}, & \text{Composition} &= z \mapsto (cz + d)^{-1} \\ z &\mapsto \left(b - \frac{ad}{c}\right)z, & \text{Composition} &= z \mapsto \frac{az + b - \frac{ad}{c}}{cz + d} \\ z &\mapsto z + a, & \text{Composition} &= z \mapsto \frac{az + b}{cz + d}. \end{aligned}$$

□

By a **circle in $\mathbb{C} \cup \{\infty\}$** we mean either an ordinary circle in $\mathbb{C}$, or the union of a line in $\mathbb{C}$ and the point $\{\infty\}$. The terminology is justified by the fact that under the stereographic projection representation of $\mathbb{C} \cup \{\infty\}$, these objects correspond to circles on the surface of the sphere.

Theorem. *A fractional linear transformation $f(z)$ sends circles in $\mathbb{C} \cup \{\infty\}$ to circles in $\mathbb{C} \cup \{\infty\}$.*

Proof. Let $E \subset \mathbb{C} \cup \{\infty\}$ be a circle, then we need to show that $f(E)$ is a circle. Writing $f(z)$ as a composition of dilations, translations, and inversions, we see that it is enough to show that circles are mapped to circles under each of these transformations. The case of dilations and translations is easy, so let's only look at the inversion $f(z) = z^{-1}$.

Suppose E is a circle in $\mathbb{C}$ with radius r and centre a , and assume that $0 \notin E$, so $|a| \neq r$. Then E is the set of points satisfying $|z - a| = r$.

$$(z - a)\overline{(z - a)} = |z - a|^2 = r^2,$$

which we write out to get

$$z\bar{z} - a\bar{z} - \bar{a}z + a\bar{a} - r^2 = 0.$$

If $a\bar{a} - r^2 \neq 0$ and for $z \neq 0$, we can divide by $z\bar{z}(a\bar{a} - r^2)$ to get

$$\begin{aligned} z^{-1}\bar{z}^{-1} - \frac{az^{-1} + \bar{a}\bar{z}^{-1}}{a\bar{a} - r^2} + \frac{1}{a\bar{a} - r^2} &= 0 \\ (z^{-1} - \frac{\bar{a}}{a\bar{a} - r^2})(\bar{z}^{-1} - \frac{a}{a\bar{a} - r^2}) &= \frac{r^2}{(a\bar{a} - r^2)^2}, \end{aligned}$$

This means that if $z \in E$, then $f(z) = z^{-1}$ lies on the circle of radius $\frac{r}{|a|^2 - r^2}$ centred in $\frac{\bar{a}}{|a|^2 - r^2}$.

A similar computation handles the remaining cases where E is a circle through 0, a line not through 0 or a line through 0, and shows that in each case $f(E)$ is either a circle in $\mathbb{C}$ or a line $\cup \{\infty\}$. $\square$

Specifying fractional linear transformations. Fractional linear transformations depend on 4 complex parameters a, b, c, d , but we have already seen that these are somewhat redundant, since $a, b, c, d \sim \lambda a, \lambda b, \lambda c, \lambda d$. This leaves 3 complex degrees of freedom, and indeed we can specify fractional linear transformations uniquely by giving 3 points in $\mathbb{C} \cup \{\infty\}$.

Theorem. *Let $z_1, z_2, z_3 \in \mathbb{C} \cup \{\infty\}$ be distinct. Then there is a unique fractional linear transformation such that*

$$f(z_1) = 0, f(z_2) = 1, f(z_3) = \infty.$$

Proof. Suppose that $z_1, z_2, z_3 \in \mathbb{C}$. Then we can let

$$f(z) = \frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1}.$$

If one of the three points z_1, z_2, z_3 is ∞ , then a similar explicit formula can be given. We omit the proof of uniqueness. $\square$

Example (MAT2410 exam 2018). Find a fractional linear transformation $f(z)$ such that $f(0) = 0$, $f(1) = 1$, and $f(2) = \infty$. What is the image of the disk $|z - 1| < 1$?

Solution: Plug in to the general formula to get

$$f(z) = \frac{z}{z - 2} \frac{1 - 2}{1} = \frac{-z}{z - 2}.$$

The image of the disk $|z - 1| < 1$ is bounded by the image of the circle $|z - 1| = 1$. We know that is a circle or a line, but since it contains $f(0) = 0$ and $f(2) = \infty$, it must be a line. Which line? Well, let's compute a third point on it: $f(1 + i) = \frac{-(1+i)}{(1+i)-2} = \frac{1+i}{i-1} = i$.¹ So then we have that the circle gets sent to the imaginary axis,

¹An alternative argument: The formula for $f(z)$ shows that it sends the real axis to the real axis. The circle $|z - 1| = 1$ is perpendicular to the real axis in $z = 0$, and since $f(z)$ is conformal, the image of the circle must be perpendicular to the image of the real axis in $f(z) = 0$. Hence the image of the circle is perpendicular to the real axis, and so must be the imaginary axis.

and the disk must get sent to either the right or left half-plane. Since $f(1) = 1$, the image is the right half-plane.