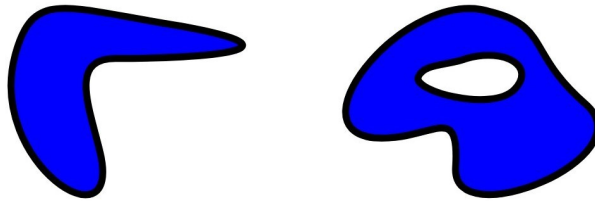


1. VIII.8 – SIMPLY CONNECTED DOMAINS

Simply connected Non-simply connected



1.1. What are simply connected domains?

Definition. Let $\gamma_0(t), \gamma_1(t), a \leq t \leq b$ be two closed paths in a domain D . A **deformation** of closed path from $\gamma_0(t)$ to $\gamma_1(t)$ in D is a family of closed paths $\gamma_s(t), a \leq t \leq b$ in D , where $0 \leq s \leq 1$, such that the map $(s, t) \mapsto \gamma_s(t)$ is a continuous function of s and t .

If there exists a deformation from $\gamma_0(t)$ to $\gamma_1(t)$, then we say that these are **deformation equivalent**.¹

Definition. Let D be a domain and let $\gamma(t)$ be a closed path in D . Then $\gamma(t)$ is **deformable to a point** if it is deformation equivalent to a constant path, $\kappa(t) = z_0$ for some $z_0 \in D$.

Definition. Let $D \subset \mathbb{C}$ be a domain. We say that D is **simply connected** if every closed path is deformable to a point.

Example. Let D be a star shaped domain. Let $z_0 \in D$ be a base point, chosen so that for every $z \in D$, the line segment

$$\{(1-s)z + sz_0 \mid s \in [0, 1]\}$$

lies in D . Now let $\gamma(t), a \leq t \leq b$ be a closed path in D with end points in z_0 . Then if we define $\gamma_s(t) = (1-s)\gamma(t) + sz_0$, we have $\gamma_0(t) = \gamma(t)$ and $\gamma_1(t) = z_0$ for all t . This shows that $\gamma(t)$ is deformable to a point, and so D is simply-connected.

Example. As special cases of simply-connected domains we find $D = \mathbb{C}$, open disks, open rectangles, and $\mathbb{C} \setminus r$, where r is a closed ray.

¹*Exercise:* If you know what an equivalence relation is, try to convince yourself that “being deformation equivalent” forms an equivalence relation on the set of closed paths in D .

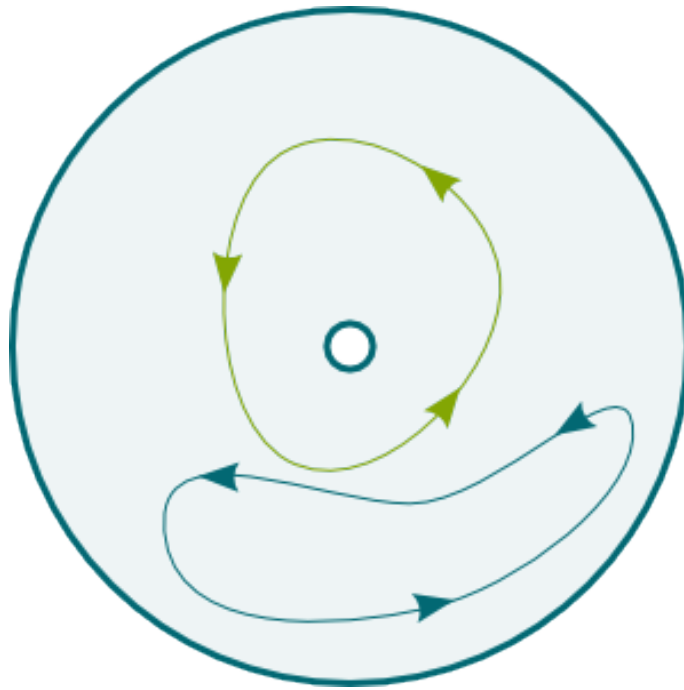


FIGURE 1. The blue loop is deformable to a point, the green loop is not.

Remark. In defining the notion of being simply connected, we haven't really used the fact that we are working with an open subset of $\mathbb{C}$. So the same definition works for any subset of $\mathbb{R}^n$, or more generally any topological space.

1.2. Analytic consequences. The following proposition is the key idea connecting the above definition to the behaviour of functions.

Proposition. *Let D be a domain, and let $Pdx + Qdy$ be a closed differential on D . Then if γ_0 and γ_1 are deformation equivalent closed paths, we have*

$$\int_{\gamma_0} Pdx + Qdy = \int_{\gamma_1} Pdx + Qdy.$$

Proof. Gamelin p. 81. □

Proposition. *If D is a simply connected domain, $Pdx + Qdy$ is a closed differential on D , and γ is a closed path in D , then*

$$\int_{\gamma} Pdx + Qdy = 0$$

Proof. Since D is simply connected, γ is deformation equivalent to $\kappa(t) = z_0$ for some point $z_0 \in D$. Thus

$$\int_{\gamma} Pdx + Qdy = \int_{\kappa} Pdx + Qdy = 0.$$

□

Proposition. Let D be a simply connected domain, $Pdx + Qdy$ be a closed differential on D and α, β be paths in D with the same start and end points. Then

$$\int_{\alpha} Pdx + Qdy = \int_{\beta} Pdx + Qdy$$

Proof. Create a closed path γ by first going along α and then back along β . Then

$$0 = \int_{\gamma} Pdx + Qdy = \int_{\alpha} Pdx + Qdy - \int_{\beta} Pdx + Qdy,$$

which gives what we wanted. □

Proposition. If D is a simply connected domain, then every closed differential on D is exact.

Proof. Let $Pdx + Qdy$ be closed, and pick a base point $(x_0, y_0) \in D$. Define $h(x, y)$ by choosing a path γ from (x_0, y_0) to (x, y) and setting

$$h(x, y) = \int_{\gamma} Pdx + Qdy.$$

By the previous proposition, this is independent of the choice of γ and so $h(x, y)$ is well-defined. Arguing as in Gamelin p. 77-78 we find that $dh = Pdx + Qdy$. □

Proposition. If D is a simply connected domain and u is a harmonic function on D , then u has a harmonic conjugate v .

Proof. Finding a harmonic conjugate v means solving the equations

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Equivalently, we want

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.$$

Since u is harmonic, this differential is closed, hence exact, hence we can find such a v . □

Proposition (Cauchy's theorem but better). If D is a simply connected domain, γ is a closed path in D and $f(z)$ is an analytic function on D , then

$$\int_{\gamma} f(z)dz = 0.$$

Proof. Since $f(z)$ is analytic, $f(z)dz$ is closed. □

Example. Let's show that $D = \mathbb{C} \setminus \{0\}$ is not simply connected. Let $\gamma(t) = e^{i\pi t}$, $t \leq 0 \leq 2\pi$. Then

$$\int_{\gamma} z^{-1} dz = 2\pi i \neq 0.$$

Therefore D does not satisfy the conclusion of the above theorem, hence is not simply connected.

Proposition. Every analytic function $f(z)$ on D has a primitive.

Proof. A complex function $F(z)$ is a primitive for $f(z)$ if $dF = f(z)dz$. Since $f(z)$ is analytic, $f(z)dz$ is closed, hence exact, hence there exists such an F . $\square$

Remark. Each of the above propositions fails if we do not assume D is simply connected: Take $D = \mathbb{C} \setminus \{0\}$, $f(z) = z^{-1}$, and for the statement about harmonic functions, $u(x, y) = \log(x^2 + y^2)$.

The complement of a simply connected domain. There is another intuition for what simplyconnectedness means, which is that the complement of the set does not have any “islands”.

Here's the first way to formalise this: It is impossible for a path in D to “loop around” (meaning have non-zero winding number with respect to) a point in $\mathbb{C} \setminus D$.

Proposition. Let D be a simply connected domain and $z_0 \in \mathbb{C} \setminus D$. If γ is a closed path in D , we have $W(\gamma, z_0) = 0$.

Proof. The function $\frac{1}{z-z_0}$ is analytic on D . We have

$$W(\gamma, z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0} = 0.$$

$\square$

The other way to make this precise is to say that the complement in $\mathbb{C} \cup \{\infty\}$ is connected.

Definition. Let $E \subseteq \mathbb{C} \cup \{\infty\}$ be a set. A function $f(z)$ on E is continuous if it is continuous as a function on $E \cap \mathbb{C}$, and additionally if $\infty \in E$, we have that

$$\lim_{n \rightarrow \infty} f(z_n) = f(\infty)$$

for every sequence z_n with $\lim_{n \rightarrow \infty} |z_n| = \infty$.

Definition. Let E be a subset of $\mathbb{C} \cup \{\infty\}$. Then E is **connected** if every continuous integer-valued function $f(x)$ is constant.

Example. An open set is connected if and only if it is a domain.

Theorem. Let $D \subseteq \mathbb{C}$ be a domain. Then the following are equivalent.

- (1) D is simply connected.

- (2) For every closed path γ in D and every analytic function $f(z)$ on D , we have $\int_{\gamma} f(z)dz = 0$.
- (3) For every $z_0 \in \mathbb{C} \setminus D$ and closed path $\gamma \in D$, the winding number $W(\gamma, z_0)$ is zero.
- (4) The complement $E = (\mathbb{C} \cup \{\infty\}) \setminus D$ is connected.

PROOF OF SOME OF THE THEOREM

(1) $\Rightarrow$ (2) : Done previously.

(2) $\Rightarrow$ (3) : Done previously.

(4) $\Rightarrow$ (3) : Let γ be a path in D . Then for $w \in E = \mathbb{C} \setminus D$, the function

$$W(\gamma, w) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - w} dz$$

is a continuous integer-valued function of w , which is 0 when $|w|$ is large enough. Hence if we define $W(\gamma, \infty) = 0$, this is a continuous function on $\overline{E}$. Since this set is connected, by assumption, we must have $W(\gamma, w) = 0$ for all w .

(3) $\Rightarrow$ (4) and (3) $\Rightarrow$ (1): See Gamelin pp. 255–257.