

CONFORMAL SELF-MAPS OF THE UNIT DISK

Question. Let $D_1 = \{z \in \mathbb{C} \mid |z| < 1\}$, and let $f(z)$ be an analytic function on D_1 defining a bijection from D_1 to D_1 . What can we say about $f(z)$?

An $f(z)$ as in the question is called a **conformal self-map of the unit disk**.

A good class of potential examples comes from looking at fractional linear transformations, that is functions of the form

$$f(z) = \frac{az + b}{cz + d}.$$

with $ad - bc \neq 0$. In order to find conformal self-maps of D_1 , we may look at the fractional linear transformations which send ∂D_1 to ∂D_1 . If one tries to spell out conditions on the coefficients a, b, c, d to make this happen, one is lead to functions of the form

$$f(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}$$

where $\theta \in [0, 2\pi)$ and $|a| < 1$.

Lemma. *The function $f(z)$ as above is a conformal self-map of the unit disk.*

Proof. We first show that $f(z)$ sends ∂D_1 to ∂D_1 . Let $|z| = 1$, and then compute

$$\begin{aligned} |f(z)|^2 &= |e^{i\theta}|^2 \frac{|z - a|^2}{|1 - \bar{a}z|^2} = \frac{(z - a)(\bar{z} - \bar{a})}{(1 - \bar{a}z)(1 - a\bar{z})} = \frac{z\bar{z} - a\bar{z} - \bar{a}z - a\bar{a}}{1 - a\bar{z} - \bar{a}z - a\bar{a}z\bar{z}} \\ &= \frac{|z|^2 - a\bar{z} - \bar{a}z - |a|^2}{1 - a\bar{z} - \bar{a}z - |a|^2|z|^2} = 1. \end{aligned}$$

Since $f(z)$ is a fractional linear transformation, it maps $\mathbb{C} \cup \{\infty\}$ bijectively to $\mathbb{C} \cup \{\infty\}$. There are in principle two pos Since

$$|f(0)| = |-ae^{i\theta}| = |a| < 1,$$

we have $f(0) \in D_1$, and so f must map D_1 bijectively to D_1 . □

In fact, every conformal self-map of unit disk is of this form, and we will show this. We start with a few lemmas.

Lemma. *As a map from D_1 to D_1 , the function*

$$f(z) = \frac{z - a}{1 - \bar{a}z}$$

has inverse

$$f^{-1}(z) = \frac{z + a}{1 + \bar{a}z}.$$

Proof. Compute $(f \circ f^{-1})(z) = z = (f^{-1} \circ f)(z)$ directly. □

Lemma (Schwarz' lemma). *Let $f(z)$ be an analytic function on D_1 such that $|f(z)| \leq 1$ for all $z \in D_1$ and $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in D_1$. If for some $z_0 \in D_1$ we have $|f(z_0)| = |z_0|$, then $f(z) = \lambda z$ for some λ with $|\lambda| = 1$.*

Proof. Let $g(z) = \frac{f(z)}{z}$. Then $g(z)$ has a removable singularity in 0, so can be extended to an analytic function on D_1 . For any disk $D_r = \{z \mid |z| < r\}$, the maximum principle asserts that $|g(z)|$ obtains its maximum on $D_r \cup \partial D_r$ on the boundary ∂D_r . On ∂D_r , we have

$$|g(z)| = |f(z)|/|z| = |f(z)|/r \leq 1/r,$$

so $|g(z)| \leq 1/r$ for all $z \in D_r$.

Now, for a given z_0 , we have $z_0 \in D_r$ for any $1 > r > |z_0|$, and so $|g(z_0)| \leq 1/r$ for any such r . Therefore $|g(z_0)| \leq \inf_{r \in (|z_0|, 1)} 1/r = 1$. It follows that

$$|f(z)| = |z||g(z)| \leq |z|.$$

If there is a $z_0 \in D_1$ such that $|f(z_0)| = |z_0|$, then $|g(z_0)| = 1$. But since $|g(z)| \leq 1$ for all z , that means that $|g(z)|$ obtains its maximum on D_1 . By the strong maximum principle, this means that $g(z)$ must be constant. Taking $g(z) = \lambda$, we have $|\lambda| = |g(z_0)| = 1$. $\square$

Lemma. *Let $f(z)$ be a conformal self-map of the unit disk, with inverse $g(z)$. Then $g(z)$ is analytic for all $z \in D_1$.*

Proof. Let $z_0 \in D_1$, and let $w_0 = f(z_0)$. If $f'(z_0) \neq 0$, then the inverse function theorem says that g is analytic in w_0 and $g'(w_0) = \frac{1}{f'(z_0)}$.

Assume there is a point $z_0 \in D_1$ such that $f'(z_0) = 0$. Since the zeros of $f'(z)$ are isolated, there is an open disk U centred in z_0 such that $f'(z) \neq 0$ for all points $z \in U \setminus \{z_0\}$. The open mapping theorem asserts that the image $f(U)$ is open, and in particular there exists an open disk V centred at w_0 such that $V \subset f(U)$.

For any point $w \in V \setminus \{w_0\}$ we have that $g(w) \in U \setminus \{z_0\}$, so $f'(g(w)) \neq 0$, and so g is analytic in w . This means that w_0 is an isolated singularity of g . But since g is bounded and continuous near w_0 , it must in fact be analytic in w_0 . $\square$

Lemma. *Let $f(z)$ be a conformal self-map of the unit disk such that $f(0) = 0$. Then $f(z) = e^{i\theta}z$ for some real θ .*

Proof. Since $f(z)$ is bijective, we have $f'(z) \neq 0$ for all z . The inverse function $g(z)$ defined by $g(f(z)) = z$ is then analytic, by the inverse function theorem. Now we have that $f(z)$ and $g(z)$ are both self-maps of D_1 satisfying $f(0) = 0$, $g(0) = 0$, so by Schwarz' lemma, we have

$$|f(z)| \leq |z| \quad |g(z)| \leq |z|$$

for all $z \in D_1$. It then follows that

$$|z| = |g(f(z))| \leq |f(z)| \leq |z|,$$

so we must have $|f(z)| = |z|$ for all z . Again by Schwarz' lemma, this means there exists a $\lambda = e^{i\theta}$ such that $f(z) = e^{i\theta}z$. $\square$

Theorem. *If $f(z)$ is a conformal self-map of the unit disk, then there exist unique $a \in D_1$ and $\theta \in [0, 2\pi)$ such that*

$$f(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}.$$

Proof. Let $f(z)$ be a conformal self-map of D_1 , let a be such that $f(a) = 0$, and let $g(z) = \frac{z+a}{1+\bar{a}z}$. Then $(f \circ g)(z)$ is another conformal self-map of D_1 , and we have

$$(f \circ g)(0) = f(g(0)) = f(a) = 0.$$

This means that $(f \circ g)(z) = e^{i\theta} z$ for some θ .

Now consider $g^{-1}(z) = \frac{z-a}{1-\bar{a}z}$. We then have

$$f(z) = (f \circ g \circ g^{-1})(z) = e^{i\theta} g^{-1}(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z},$$

which is what we wanted.

For uniqueness, notice that a is fully determined by $f(a) = 0$, and θ is then determined by $f(1) = e^{i\theta} \frac{1-a}{1-\bar{a}}$. $\square$