



UiO : **Department of Mathematics**
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Complex numbers

- Complex numbers $z = x + iy$ with x, y real.
- Addition $z_1 + z_2$, multiplication $z_1 z_2$, division z_1/z_2 if $z_2 \neq 0$.
- **Modulus** $|z| = \sqrt{x^2 + y^2}$, length of z as vector
- **Argument** $\arg(z)$: angle between positive real axis and z .
- **Polar form** $z = re^{i\theta}$ with $r = |z|$ and $\theta = \arg(z)$

$$r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

- **Triangle inequalities:** If z, w are complex numbers, then

$$|z| - |w| \leq |z + w| \leq |z| + |w|$$

Complex functions

Function $f(z)$ of a complex number z with complex values.

Simple examples

- $f(z) = z, z^n, \bar{z}, \operatorname{Re}(z), \operatorname{Im}(z), |z|$
- $f(z) = e^z = e^x(\cos y + i \sin y)$
- $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$ and $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$

Argument function

- $\arg(z)$ is only defined up to multiples of 2π .
- *E.g.* $\arg(1 + i) = \pi/4, \pi/4 + 2\pi, \pi/4 - 2\pi, \dots$
- **Principal value** of $\arg(z)$ is

$\text{Arg}(z) = \text{choice of argument so that } \text{Arg}(z) \in (-\pi, \pi].$

Logarithm function

- If $z \neq 0$ and $\log(z)$ is such that $e^{\log(z)} = z$, then this does not determine $\log(z)$ uniquely, only up to multiples of $2\pi i$.
- *E.g.* $\log(i) = i\pi/2, i\pi/2 + 2\pi i, i\pi/2 + 4\pi i, \dots$
- **Branch** of logarithm function is a function $f(z)$ giving continuous choice of value of $\log(z)$.
- There is no branch which is defined on all of $\mathbb{C} \setminus \{0\}$.
- **Principal branch** of logarithm function is defined as

$$\text{Log}(z) = \log(|z|) + i\text{Arg}(z), \quad z \in \mathbb{C} \setminus \{a \in \mathbb{R} \mid a \leq 0\}.$$

Root functions

- Let $n \geq 2$ be an integer and let $z \neq 0$. If $z^{\frac{1}{n}}$ is such that $(z^{\frac{1}{n}})^n = z$, then this does not determine $z^{\frac{1}{n}}$ uniquely, since we can multiply by $e^{\frac{2\pi i}{n}}$ to get a new n -th root of z .

- *E.g.*

$$(-1)^{1/2} = i, -i.$$

- A **branch** of the n -th root function is a function $f(z)$ giving continuous choice of values of $z^{\frac{1}{n}}$.
- *E.g.* $f(z) = e^{\frac{\text{Log}(z)}{n}}$ is a branch of $z^{\frac{1}{n}}$, defined for $z \in \mathbb{C} \setminus \{a \in \mathbb{R} \mid a \leq 0\}$.

Subsets of $\mathbb{C}$

A subset $E \subset \mathbb{C}$ is

- **open** if for every $z_0 \in E$, there is a small disk centred on z_0 contained in E .
- **closed** if every sequence $z_1, z_2, \dots \in E$ such that $\lim_{n \rightarrow \infty} z_n = z$ must have $z \in E$.
- A **domain** if it is open and any two points can be connected by a polygonal path.
- **simply connected** if every closed path in E can be continuously contracted to a point (e.g. open squares, disks, $\mathbb{C}$ itself, star-shaped domains).

Differentiable and analytic functions

- A complex function $f(z)$ is **differentiable** in z_0 if the limit

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, and if so $f'(z_0)$ is defined to be this limit.

This means that for *every* sequence $z_1, z_2, \dots$ such that $z_n \rightarrow z_0$, we have $\frac{f(z_n) - f(z_0)}{z_n - z_0} \rightarrow f'(z_0)$.

- A function $f(z)$ is **analytic** on a domain D if it is differentiable in each point of D and $f'(z)$ is continuous.¹
- Basic analytic functions: z^n , e^z , $\cos(z)$, $\sin(z)$, $\text{Log}(z)$
- Not analytic: $\text{Re}(z)$, $\text{Im}(z)$, $|z|$, $\bar{z}$, $\text{Arg}(z)$

¹“ $f'(z)$ is continuous” is automatically true if f is differentiable in each point (Goursat's theorem).

Creating analytic functions

If $f(z)$ and $g(z)$ are analytic, then so are

- $f(z) + g(z)$, with derivative $f'(z) + g'(z)$
- $f(z)g(z)$, with derivative $f'(z)g(z) + f(z)g'(z)$
- $f(z)/g(z)$, where $g(z) \neq 0$, with derivative

$$\frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}.$$

- $(f \circ g)(z)$, with (“the chain rule”)

$$(f \circ g)'(z) = f'(g(z))g'(z)$$

The Cauchy–Riemann equations

If $f(z) = u(x, y) + iv(x, y)$ is analytic, then the limit

$$\lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

exists. Taking limit along $\Delta z = \Delta x$ real and $\Delta z = i\Delta y$ imaginary gives two expressions for $f'(z) \rightsquigarrow$ **Cauchy–Riemann equations**

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}.\end{aligned}$$

Conversely, if these equations hold and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous, then $f(z)$ is analytic.

Proving a function is analytic

Three options:

- 1 Show directly that the limit

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

- 2 Use Cauchy–Riemann equations.
- 3 Show that the function is expressible through other functions using the rules (e.g. $e^{\sin(z)-z^2} + z^3$).

Harmonic functions

- A **harmonic function** is a real function (with continuous second derivatives) $h(x, y)$ satisfying **Laplace's equation**:

$$\frac{\partial^2 h}{(\partial x)^2} + \frac{\partial^2 h}{(\partial y)^2} = 0.$$

- If $f(z) = u(x, y) + iv(x, y)$ is analytic, then from the CR equations we get that u and v are harmonic.
- If $u(x, y)$ is a harmonic function on a domain D , then a **harmonic conjugate** of u is a function v such that $u + iv$ is analytic, equivalently such that the Cauchy–Riemann equations hold for u, v .

Finding a harmonic conjugate

TO BE ADDED

The mean value property

If $h(x, y)$ is harmonic, then it is “well balanced” in the following way:
For any point (x_0, y_0) , the average value of h on a circle with centre (x_0, y_0) equals $h(x_0, y_0)$.

Mean value property

If $h(x, y)$ is harmonic on a domain D containing the disk

$$D_0 = \{(x, y) \mid |(x, y) - (x_0, y_0)| \leq r\},$$

then

$$\frac{1}{2\pi} \int_0^{2\pi} h(x_0 + r \cos \theta, y_0 + r \sin \theta) d\theta = h(x_0, y_0).$$

The maximum principle

Let D be a domain.

Maximum principle, harmonic functions

Let $h(x, y)$ be a (real or complex) harmonic function on D such that $|h(x, y)| \leq M$ for all $(x, y) \in D$ and some number M . If $h(x_0, y_0) = M$ for some $(x_0, y_0) \in D$, then $h(x, y) = M$ for all $(x, y) \in D$.

Maximum principle, complex version

Let $f(z)$ be an analytic function on D such that $|f(z)| \leq M$ for all $z \in D$ and some number M . If $|f(z_0)| = M$ for some $z_0 \in D$, then $f(z) = f(z_0)$ for all $z \in D$.

One more maximum principle

Let D be a *bounded* domain.

Maximum principle

If $f(z)$ is a function which is analytic on D and continuous on $D \cup \partial D$, then there is a $z_0 \in \partial D$ such that

$$|f(z)| \leq |f(z_0)|$$

for all $z \in D \cup \partial D$.

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Revision Chapters 1, 2, some of 3

