



UiO : **Department of Mathematics**
University of Oslo

MAT2410 Autumn 2022

Revision – Integrals and power series

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Line integrals

- **Path:** Function

$$\gamma(t) = (x(t), y(t)) \leftrightarrow x(t) + iy(t), a \leq t \leq b.$$

Usually assumed *piecewise smooth*: $x(t), y(t)$ infinitely differentiable except in a finite number of points.

- A **differential** is an expression $P(x, y)dx + Q(x, y)dy$, where $P(x, y), Q(x, y)$ are (real or complex) functions.
- The **line integral** of $Pdx + Qdy$ along γ is

$$\int_{\gamma} Pdx + Qdy = \int_a^b P(\gamma(t))x'(t) + Q(\gamma(t))y'(t)dt.$$

Green's theorem

If D is a bounded domain with piecewise smooth boundary ∂D ,

$$\int_{\partial D} Pdx + Qdy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

We orient ∂D as a path (or collection of paths) in such a way that D is on the left.

Closed differentials

- A differential $Pdx + Qdy$ defined on a domain D is **closed** if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

- Equivalently, for every bounded domain $E \subset D$ with piecewise smooth boundary $\partial E \subset D$, we have

$$\int_{\partial E} Pdx + Qdy = 0.$$

- If $Pdx + Qdy$ is closed, and the path γ_1 can be continuously deformed to γ_2 in D (keeping the end-points fixed), then

$$\int_{\gamma_1} Pdx + Qdy = \int_{\gamma_2} Pdx + Qdy.$$

Exact differentials

- The **differential** of a function $h(x, y)$ is given by

$$dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy$$

- A differential $Pdx + Qdy$ is **exact** if there exists a h such that $dh = Pdx + Qdy$.
- If $Pdx + Qdy = dh$, and $\gamma(t)$, $a \leq t \leq b$ a path, then

$$\int_{\gamma} Pdx + Qdy = h(\gamma(b)) - h(\gamma(a)).$$

- $Pdx + Qdy$ exact $\Leftrightarrow \int_{\gamma} Pdx + Qdy$ always determined by the end points of $\gamma \Leftrightarrow \int_{\gamma} Pdx + Qdy = 0$ for all closed paths γ .

Closed vs. exact

- $Pdx + Qdy$ exact $\Rightarrow Pdx + Qdy$ closed, but not conversely.
- On a *simply connected* domain D , a differential is exact if and only if it is closed. E.g: On $\mathbb{C}$, on a disk, a square, on any star-shaped domain.

Complex line integrals

If $f(z)$ is a complex function and $\gamma(t)$ a path, then set $dz = dx + idy$, and define

$$\int_{\gamma} f(z) dz = \int_{\gamma} f(z) dx + i f(z) dy.$$

Equivalently, with $\gamma(t) = x(t) + iy(t)$, get

$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt.$$

The ML estimate

Let $f(z)$ be a complex function and $\gamma(t) = x(t) + iy(t)$, $a \leq t \leq b$ a path. Let L be the path length of γ , defined by

$$L = \int_a^b |\gamma'(t)| dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt.$$

Let M be a real number such that $|f(\gamma(t))| \leq M$ for all t . Then

$$\left| \int_{\gamma} f(z) dz \right| \leq ML$$

Differentials and analytic functions

- If $f(z)$ is analytic, then $f(z)dz$ is *closed*.
- A **primitive** for $f(z)$ is an analytic function $F(z)$ such that $F'(z) = f(z)$.
- E.g: $f(z) = z^n, n \geq 0$ has primitive $\frac{1}{n+1}z^{n+1}$.
 $f(z) = \frac{1}{z}$ does not have primitive on $D = \mathbb{C} \setminus \{0\}$.
- We have $f(z)dz$ exact $\Leftrightarrow f(z)$ has a primitive $F(z)$ with $dF = f(z)dz$.
- **Fundamental theorem of calculus, complex version:** If $F(z)$ is a primitive for $f(z)$, then $\int_{\gamma} f(z)dz = F(\gamma(b)) - F(\gamma(a))$.

Cauchy's theorems

Let D be a bounded domain with piecewise smooth boundary, and let $f(z)$ be an analytic function. Then $f'(z)$ is itself analytic, and we have, for $z_0 \in D$, that

- Cauchy's theorem: $\int_{\partial D} f(z) dz = 0$
- Cauchy's integral formula: $\int_{\partial D} \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0).$
- Cauchy's integral formula for derivatives:
$$\int_{\partial D} \frac{f(z)}{(z-z_0)^{k+1}} dz = \frac{2\pi i}{k!} f^{(k)}(z_0).$$

Power series

- A **power series** is an infinite sum of the form

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

where a_k, z, z_0 are complex numbers.

- Fixing the a_k and z_0 , where does this converge and define a function of z ?
- There is a **radius of convergence** $R \in [0, \infty]$ such that the series converges *absolutely* on $\{z \in \mathbb{C} \mid |z - z_0| < R\}$, and diverges on $\{z \in \mathbb{C} \mid |z - z_0| > R\}$.
- Compute R as

$$\sup\{r \in \mathbb{R} \mid \{r^k |a_k|\} \text{ is a bounded sequence}\}.$$

Power series expansion of an analytic function

Let $f(z)$ be an analytic function on a domain D , and let $z_0 \in D$.

- Then we have a power series expansion

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k = f(z),$$

which is valid “near z_0 ”.

- Compute coefficients by

$$a_k = \frac{f^{(k)}(z_0)}{k!}.$$

- Radius of convergence is the radius of the biggest disk D_r centred in z_0 such that $f(z)$ can be extended to an analytic function on D_r .¹

¹See Gamelin for precise statement.

Working with power series, I

If $f(z)$ is analytic near z_0 with power series expansion $\sum_{k=0}^{\infty} a_k(z - z_0)^k$, then get a power series expansion

$$f'(z) = \sum_{k=0}^{\infty} k a_k (z - z_0)^{k-1},$$

and can define a primitive $F(z)$ near z_0 by

$$F(z) = C + \sum_{k=0}^{\infty} \frac{a_k}{k+1} (z - z_0)^{k+1}, \quad C \in \mathbb{C}.$$

Working with power series, II

If $f(z)$ and $g(z)$ are analytic functions with power series expansions

$f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$ and $g(z) = \sum_{k=0}^{\infty} b_k(z - z_0)^k$, then

- $f(z) + g(z) = \sum_{k=0}^{\infty} (a_k + b_k)(z - z_0)^k$
- $f(z)g(z) = \sum_{k=0}^{\infty} c_k(z - z_0)^k$, with $c_k = \sum_{i=0}^k a_i b_{k-i}$.

Working with power series, III

Assume $g(z_0) = 1$. Can compute power series expansion of $1/g(z)$ by

$$\begin{aligned}\frac{1}{g(z)} &= \frac{1}{1 + \sum_{k=1}^{\infty} b_k(z - z_0)^k} = \frac{1}{1 - (\sum_{k=1}^{\infty} -b_k(z - z_0)^k)} \\ &= \sum_{j=0}^{\infty} \left(\sum_{k=1}^{\infty} -b_k(z - z_0)^k \right)^j,\end{aligned}$$

which gives a well defined power series

$$\frac{1}{g(z)} = 1 - b_1(z - z_0) + (b_1^2 - b_2)(z - z_0)^2 + \dots.$$

Using this can compute expansion of $f(z)/g(z)$ as $f(z) \left(\frac{1}{g(z)} \right)$.

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