



UiO : **Department of Mathematics**
University of Oslo

MAT2410 Autumn 2022

Revision – Singularities and residue integrals

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Isolated zeros

- An analytic function $f(z)$ which is not constant 0 has only *isolated* zeros.
- The **order** of a zero z_0 is n if the power series expansion of $f(z)$ at z_0 is

$$f(z) = a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \cdots, \quad (a_n \neq 0)$$

- Equivalently: Order is n if

$$0 = f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0), 0 \neq f^{(n)}(z_0).$$

- Equivalently: Order is n if we can write

$$f(z) = (z - z_0)^n h(z),$$

with $h(z)$ analytic in z_0 and $h(z_0) \neq 0$.

Isolated zeros, more

Two examples

- $f(z) = z^n$ has a zero of order n in 0.
- $f(z) = \sin(z)$ has a zero of order 1 in 0, since

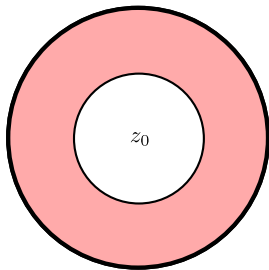
$$f(z) = 0, f'(z) = \cos(0) = 1 \neq 0.$$

Annulus

An **annulus** centred at z_0 is a set of the form

$$\{z \in \mathbb{C} \mid \rho < |z - z_0| < \sigma\}$$

for some chosen numbers $0 \leq \rho < \sigma \leq \infty$.



Laurent series

- A **Laurent series** is a series of the form.

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k,$$

- Given an annulus $D = \{\rho < |z - z_0| < \sigma\}$ and a function $f(z)$ analytic on D , there is a unique Laurent series representation of f such that the sum converges absolutely on D .
- Coefficients determined by

$$a_k = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z - z_0)^{k+1}} dz,$$

for any $r \in (\rho, \sigma)$.

Laurent series, example

The function $f(z) = \frac{1}{1-z}$ is analytic on the disk

$$D_1 = \{z \mid |z| < 1\}$$

and the annulus

$$D_2 = \{z \mid 1 < |z|\}.$$

It's Laurent series representation on D_1 is

$$\frac{1}{1-z} = 1 + z + z^2 + \dots$$

On D_2 , this diverges. We therefore instead take the Laurent series

$$\frac{1}{1-z} = z^{-1} \frac{1}{z^{-1} - 1} = -z^{-1} - z^{-2} - z^{-3} - \dots$$

Isolated singularities

A function $f(z)$ has an **isolated singularity** at a point z_0 if there is an $r > 0$ such that $f(z)$ is analytic on $\{z \mid 0 < |z - z_0| < r\}$, and either undefined or not analytic in z_0 .

An isolated singularity is one of three kinds:

- 1 **Removable singularity**
- 2 **Pole**
- 3 **Essential singularity**

Classification is based on the Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k$$

Removable singularities

- Let $f(z)$ have an isolated singularity at z_0 and Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k$$

near z_0 . Then $f(z)$ has a removable singularity at z_0 if $a_k = 0$ for all $k < 0$.

- If the singularity is removable, can redefine $f(z)$ by setting

$$f(z_0) = \lim_{z \rightarrow z_0} f(z) = a_0,$$

to make $f(z)$ analytic at z_0 .

- **Riemann's theorem:** The singularity is removable if $f(z)$ is bounded near z_0 .

Removable singularity, example

Let $f(z) = \frac{\sin(z)}{z}$. The Laurent expansion of $f(z)$ near z_0 is

$$f(z) = \frac{1}{z} \sin(z) = \frac{1}{z} \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots,$$

hence the singularity at 0 is removable. We can redefine $f(0) = 1$ to make $f(z)$ analytic at 0.

Poles

- Let $f(z)$ have an isolated singularity at z_0 . Then $f(z)$ has a **pole of order n** if the Laurent expansion near z_0 is

$$f(z) = a_{-n}(z - z_0)^{-n} + a_{-n+1}(z - z_0)^{-n+1} + \dots$$

- The function $f(z)$ has a pole at z_0 if and only if $\lim_{z \rightarrow z_0} |f(z)| = \infty$.

Computing pole and zero orders

For a function $f(z)$ analytic near z_0 which does not have an essential singularity at z_0 , define

$$\text{ord}_{z_0}(f) = \begin{cases} n & \text{if } f(z) \text{ has a zero of order } n \text{ in } z_0 \\ 0 & \text{if } f(z) \text{ is analytic in } z_0 \text{ with } f(z_0) \neq 0 \\ -n & \text{if } f(z) \text{ has a pole of order } n \text{ in } z_0 \end{cases}$$

Then given two such functions f and g , we have

$$\begin{aligned}\text{ord}_{z_0}(fg) &= \text{ord}_{z_0}(f) + \text{ord}_{z_0}(g) \\ \text{ord}_{z_0}(f/g) &= \text{ord}_{z_0}(f) - \text{ord}_{z_0}(g)\end{aligned}$$

Example: $f(z) = \frac{z}{\sin(z)^2}$: Here z has zero of order 1 in 0, $\sin(z)^2$ has a zero of order 2, hence $f(z)$ has a pole of order 1:

$$\text{ord}_0(f) = \text{ord}_0(z) - \text{ord}_0(\sin(z)^2) = 1 - 2 = -1.$$

Essential singularities

- A function $f(z)$ has an essential singularity at z_0 if the Laurent expansion at z_0

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

has infinitely many non-zero a_k with $k < 0$.

- *Example:* $f(z) = e^{1/z} = \sum_{k=0}^{\infty} \frac{z^{-k}}{k!}$ has an essential singularity at 0.
- **Casorati–Weierstrass:** If $f(z)$ has an essential singularity at z_0 , then for every complex number c , we can find a sequence $z_1, z_2, \dots$ with $\lim_{n \rightarrow \infty} z_n = z_0$, such that

$$\lim_{n \rightarrow \infty} f(z_n) = c.$$

Everything at ∞

- We define what it means for a function $f(z)$ to have a “zero of order n ”/“pole of order n ”/“essential singularity” at ∞ as follows. Let $g(z) = f(1/z)$. Then we say $f(z)$ has property P at ∞ if the function $g(z)$ has property P at 0 .
- *Example:* Taking $f(z) = e^z \rightsquigarrow g(z) = e^{1/z}$, which has an essential singularity at 0 . Hence $f(z)$ has an essential singularity at ∞ .

Partial fractions decomposition

Let $f(z) = \frac{P(z)}{Q(z)}$ be a rational function. By the fundamental theorem of algebra, we can write

$$f(z) = c \frac{\prod_{i=1}^k (z - z_i)^{m_i}}{\prod_{j=1}^l (z - \zeta_j)^{n_j}}, \quad c, z_i, \zeta_j \in \mathbb{C}$$

where the $z_1, \dots, z_k, \zeta_1, \dots, \zeta_l$ are all distinct. The **partial fractions decomposition** of $f(z)$ is a presentation

$$f(z) = \sum_{j=0}^l P_j(z),$$

where $P_0(z)$ is a polynomial, and $P_j(z)$ has the form

$$P_j(z) = \sum_{a=1}^{n_j} \frac{c_{aj}}{(z - \zeta_j)^a}, \quad c_{aj} \in \mathbb{C}$$

The residue theorem

If $f(z)$ has an isolated singularity at z_0 , with Laurent series expansion near z_0 given

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k,$$

then the **residue** of $f(z)$ at z_0 is

$$\operatorname{Res}[f(z), z_0] = a_{-1}.$$

The residue theorem: Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is analytic on $D \cup \partial D$, except for isolated singularities at $z_1, \dots, z_n \in D$, then

$$\int_{\partial D} f(z) dz = 2\pi i \sum_{j=1}^n \operatorname{Res}[f(z), z_j].$$

Computing residues

- In simple cases, can compute $\text{Res}[f(z), z_0]$ by computing Laurent series of $f(z)$ in z_0 .
- A useful rule: If $f(z)$ has a pole of order n in z_0 , then let

$$h(z) = (z - z_0)^n f(z).$$

We then have

$$\text{Res}[f(z), z_0] = \lim_{z \rightarrow z_0} \frac{h^{(n-1)}(z)}{(n-1)!}$$

- Can use **L'Hôpital's rule**: If $f(z)$ and $g(z)$ are analytic at z_0 with $f(z_0) = g(z_0) = 0$, then

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \lim_{z \rightarrow z_0} \frac{f'(z)}{g'(z)}.$$

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