



UiO : **Department of Mathematics**
University of Oslo

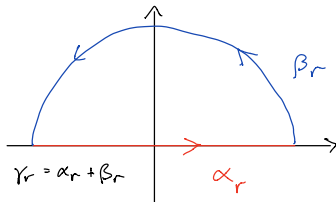
MAT2410 Autumn 2022

Revision – The remaining bits

28.11.2022

Real integrals, I

Let $f(z)$ be a function on $\mathbb{C}$, want to compute $\int_{-\infty}^{\infty} f(x)dx$.



$$\int_{-\infty}^{\infty} f(x)dx = \int_{\gamma_r} f(z)dz = \int_{\alpha_r} f(z)dz + \int_{\beta_r} f(z)dz.$$

For certain functions

- $\lim_{r \rightarrow \infty} \int_{\beta_r} f(z)dz = 0$, so
- $\int_{-\infty}^{\infty} f(x)dx = \lim_{r \rightarrow \infty} \int_{\gamma_r} f(z)dz$, and
- $\lim_{r \rightarrow \infty} \int_{\gamma_r} f(z)dz$ is computable via the residue theorem.

Real integrals, I, examples

Typical cases where this works

- Rational functions $f(z) = \frac{P(z)}{Q(z)}$ with $\deg P \leq \deg Q - 2$.
- Functions $f(z) = \frac{P(z)}{Q(z)} T(z)$ where $T(z)$ is a trigonometric function and $P(z), Q(z)$ are polynomials.

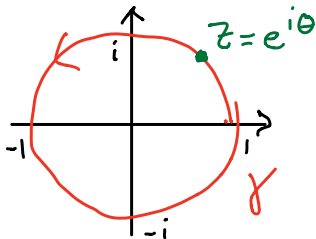
The simplest example

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi.$$

Real integrals, II

Certain definite integrals can also be reinterpreted as complex line integrals around the unit disk. *E.g.*

$$\int_0^{2\pi} \frac{1}{2 + \cos \theta} d\theta = \int_0^{2\pi} \frac{1}{2 + \frac{e^{i\theta} + e^{-i\theta}}{2}} d\theta = \int_{\gamma} \frac{-iz^{-1}}{2 + \frac{z+z^{-1}}{2}} dz.$$



This line integral can be handled by the residue theorem.

The argument principle

A function $f(z)$ on a domain D is **meromorphic** if it is analytic except that it may have isolated singularities, all of which are poles.

Let

- D be a bounded domain
- $f(z)$ be a meromorphic function which is analytic and non-zero on ∂D .

Let N_0 be the number of zeros of $f(z)$ and N_∞ the number of poles, both counted with order. Then

$$N_0 - N_\infty = \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)} dz.$$

The argument principle, example

Let's prove the fundamental theorem of algebra. Let

$f(z) = c_n z^n + c_{n-1} z^{n-1} + \dots + c_0$, and let D_r be a disk centred in 0 of radius r . When $|z| = r$ and $r \gg 0$, we have

$$\frac{f'(z)}{f(z)} = \frac{nc_n z^{n-1} + \dots + c_1}{c_n z^n + \dots + c_0} = \frac{nc_n + \dots + c_1 z^{-n+1}}{c_n + \dots + c_0 z^{-n}} z^{-1} \approx n z^{-1}.$$

So the number of zeros of $f(z)$ in D_r is

$$N_0 = \frac{1}{2\pi i} \int_{\partial D_r} \frac{f'(z)}{f(z)} dz \approx \frac{1}{2\pi i} \int_{\partial D_r} n z^{-1} dz = n.$$

One can check that $\approx$ here means $|N_0 - n| < 1$. Since N_0 is an integer, this means $N_0 = n$.

Rouché's theorem

or, “Perturbing an analytic function slightly does not change its number of zeros in a given domain”:

Rouché's theorem

Let

- D be a bounded domain
- $f(z)$ be a function analytic on $D \cup \partial D$.
- $g(z)$ a function analytic on $D \cup \partial D$, with

$$|g(z)| < |f(z)| \quad \text{for all } z \in \partial D.$$

Then the number of zeros of $f(z)$ on D equals the number of zeros of $f(z) + g(z)$ on D .

Rouché's theorem, example

From Gamelin: Consider

$$p(z) = z^6 + 9z^4 + z^3 + 2z + 4.$$

What is the number of zeros in $D = \{z \in \mathbb{C} \mid |z| < 1\}$?

Take

$$f(z) = 9z^4$$

$$g(z) = z^6 + z^3 + 2z + 4$$

When $|z| = 1$, we have

$$|g(z)| \leq |z^6| + |z^3| + |2z| + |4| = 1 + 1 + 2 + 4 = 8 < 9 = |f(z)|.$$

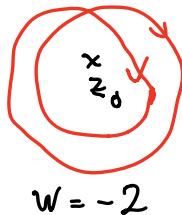
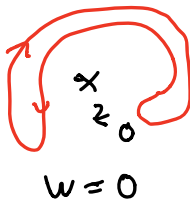
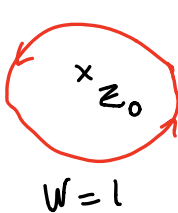
So no. of zeros of $p(z) = f(z) + g(z)$ equals no. of zeros of $f(z)$, i.e. 4.

Winding numbers

Let $z_0 \in \mathbb{C}$, $\gamma(t)$ be a closed piecewise smooth path such that $\gamma(t) \neq z_0$ for all t .

The **winding number** of γ around z_0 is

$$W(\gamma, z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0}.$$



Simply connected domains

A domain D is **simply connected** if every closed path γ can be continuously deformed to a constant path.

Examples

D a disk, a square, $\mathbb{C}$, $\mathbb{C} \setminus \{a \in \mathbb{R} \mid a \leq 0\}$.

Non-examples

$D = \mathbb{C} \setminus \{0\}$, or any other annulus, is not simply connected.

On a simply connected domain

- Every closed differential is exact.
- Every analytic function $f(z)$ has a primitive.

Conformal maps

A function $f(z)$ is **conformal** if it preserves angles.



Theorem

An analytic function $f(z)$ such that $f'(z_0) \neq 0$ is conformal at z_0 .

Fractional linear transformations

A function of the form

$$f(z) = \frac{az + b}{cz + d}$$

with $ad - cb \neq 0$ is called a **fractional linear transformation**.

- $f(z)$ is uniquely determined by specifying value in 3 points

$$f(z_1) = w_1, f(z_2) = w_2, f(z_3) = w_3$$

- $f(z)$ defines a *bijection* $\mathbb{C} \cup \{\infty\} \leftrightarrow \mathbb{C} \cup \{\infty\}$.
- If $E \subset \mathbb{C}$ is a line or a circle, then $f(E)$ is a line or a circle.

Conformal self maps of the unit disk

Let $D_1 = \{z \in \mathbb{C} \mid |z| < 1\}$. A **conformal self-map of the unit disk** is a function $f(z)$ which is analytic on D_1 and defines a bijection $D_1 \leftrightarrow D_1$.

Theorem

A conformal self-map of the unit disk can be written uniquely as

$$f(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}.$$

with $|a| < 1$ and $\theta \in [0, 2\pi)$.

Fluid mechanics

A vector field $\mathbf{V} = (P, Q)$ defined on a domain $D \subset \mathbb{C}$ represents a fluid flow. We assume it is

■ incompressible: $\nabla \cdot \mathbf{V} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0$.

■ irrotational: $\nabla \times \mathbf{V} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \Leftrightarrow Pdx + Qdy$ is closed.

A **complex velocity potential** for $\mathbf{V}$ is an analytic function $f(z) = \varphi(z) + i\psi(z)$ such that

$$d\varphi = Pdx + Qdy.$$

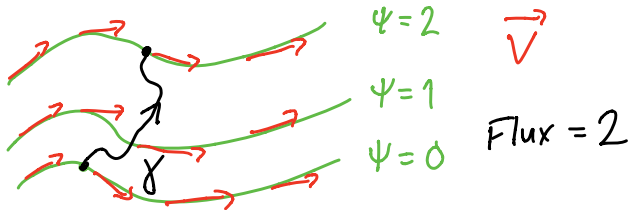
This always exists when D is simply connected.

Benefits of having a complex velocity potential

If $f(z) = \phi(z) + i\psi(z)$ is a complex velocity potential for $\mathbf{V}$:

- The flow lines of $\mathbf{V}$ are the level sets of $\psi(z)$.
- The flux across a path $\gamma(t)$, $a \leq t \leq b$ is

$$\begin{aligned}\text{Flux} &\stackrel{\text{def}}{=} \int_a^b P(\gamma(t))y'(t) - Q(\gamma(t))x'(t)dt \\ &= \psi(\gamma(b)) - \psi(\gamma(a)).\end{aligned}$$



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