

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT2410 — Introduction to Complex Analysis

Day of examination: Tuesday 28. November 2023

Examination hours: 09:00–13:00

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

For each question you may use results from the preceding questions even if you haven't answered them.

Problem 1 (weight 10%)

We consider the complex number z defined by:

$$z = \frac{1}{2} + \frac{\sqrt{3}}{2}i. \quad (1)$$

Provide a geometric interpretation of z and deduce that $z^6 = 1$.

Problem 2 (weight 10%)

Define the function $f : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ by, for $z \neq 0$, $f(z) = 1/z$. Justify that f does not have a primitive on its domain of definition.

Problem 3 (weight 10%)

Let $D \subseteq \mathbb{C}$ be a domain. Let $f : D \rightarrow \mathbb{C}$ be an analytic function. Suppose also that the conjugate $\bar{f}$ is analytic. Prove that f is constant.

Problem 4 (weight 20%)

a (weight 10%)

Let the complex function f be defined by:

$$f(z) = \frac{1}{(z^2 + 1)^2}. \quad (2)$$

Determine the poles of f , their orders and the corresponding residues.

(Continued on page 2.)

b (weight 10%)

Compute the integral:

$$\int_{-\infty}^{+\infty} f(z) dz. \quad (3)$$

Problem 5 (weight 10%)

Let P be the polynomial defined by $P(z) = z^5 + 3z - 1$. Justify that P has one and only one root in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$.

Problem 6 (weight 20%)

We define a function $u : \mathbb{C} \rightarrow \mathbb{R}$ by, for $x, y \in \mathbb{R}$:

$$u(x + iy) = e^{-x} \sin(y). \quad (4)$$

a (weight 10%)

Justify that u has a harmonic conjugate and compute one.

b (weight 10%)

What is the average of u on the circle:

$$\partial B(i\pi/2, \pi/2) = \{z \in \mathbb{C} : |z - i\pi/2| = \pi/2\} \quad (5)$$

Problem 7 (weight 20%)

For $z \in \mathbb{C} \setminus \{1\}$ we define:

$$f(z) = \exp\left(\frac{1}{1-z}\right). \quad (6)$$

We also introduce the real part of f as the function u defined by:

$$u(z) = \Re f(z). \quad (7)$$

a (weight 10%)

We let D be the open unit disc:

$$D = \{z \in \mathbb{C} : |z| < 1\}. \quad (8)$$

Justify that u is harmonic on D and that u is also unbounded on D .

b (weight 10%)

We let C be unit circle:

$$C = \{z \in \mathbb{C} : |z| = 1\}. \quad (9)$$

Check that for $z \in C \setminus \{1\}$ we have $|f(z)| = \exp(1/2)$. In particular u is bounded on $C \setminus \{1\}$ but not on D . Why is there no contradiction with the maximum principle?

THE END